TRIVIAL SOURCE CHARACTER TABLES OF FROBENIUS GROUPS OF TYPE $(C_p \times C_p) \rtimes H$

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Dedicated to the memory of Richard Parker

Abstract. Let p be a prime number. We compute the trivial source character tables of finite Frobenius groups G with an abelian Frobenius complement H and an elementary abelian Frobenius kernel of order p^2. More precisely, we deal with infinite families of such groups which occur in the two extremal cases for the fusion of p-subgroups: the case in which there exists exactly one G-conjugacy class of non-trivial cyclic p-subgroups, and the case in which there exist exactly $p+1$ distinct G-conjugacy classes of non-trivial cyclic p-subgroups.

1. Introduction

Let G be a finite group. Let p be a prime number dividing the order of G and let k be an algebraically closed field of characteristic p. Permutation kG-modules and their direct summands – called *p-permutation modules* or also *trivial source modules* – are omnipresent in the modular representation theory of finite groups. They are, for example, elementary building blocks for the construction and for the understanding of different categorical equivalences between block algebras, such as splendid Rickard equivalences, p-permutation equivalences, source-algebra equivalences, or Morita equivalences with endo-permutation source. A deep understanding of the structure of these modules is therefore essential.

In this manuscript, we go back to ideas of Benson and Parker developed in [BP84]. Any trivial source kG-module can be lifted to characteristic zero and affords a well-defined ordinary character, which contains essential information about its structure. The *trivial source character table* $\mathrm{Triv}_p(G)$ *of* G *at the prime* p collects this information in a table; it is the *species table* or *representation table* of the trivial source ring in the sense of [BP84, Ben84, Ben98]. More precisely, it provides us with information about the character values of all the indecomposable trivial source kG-modules and their Brauer quotients at all p'-conjugacy classes. See Subsection 2.3 for a precise definition.

The present article is in fact part of a program aiming at gathering information about trivial source modules of small finite groups and their associated *trivial source character tables* in a database [BFLP24]. Isolated examples – calculated by Benson, and Lux and Pahlings – can be found in [Ben84, Appendix] and [LP10, §4.10]. More recently, the first author, as part of his doctoral thesis [Böh24], developed GAP4 [GAP] and MAGMA [BCP97] algorithms, which could be used to compute the trivial source character tables of finite groups of order less than 100, as well as the trivial source character tables of various small (non-abelian) quasi-simple groups. The latter algorithms rely, in particular, on the MeatAxe algorithm, first introduced

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by R. Parker. Meanwhile, in [BFL22] and [FL23] the authors and Farrell computed *generic* trivial source character tables for the groups $\mathrm{SL}_2(q)$ and $\mathrm{PSL}_2(q)$ in cross characteristic, using the generic character tables of these groups and theoretical methods involving block theory.

Using the data produced in [BFLP24] we identified some interesting families of finite groups, of which we can calculate the trivial source character tables from a purely theoretical point of view. In this regard, the main results of this article consist in the calculation of the trivial source character tables at p of the following infinite families of Frobenius groups with an abelian Frobenius complement H and elementary abelian Frobenius kernel of rank 2 :

(I) the family of all metabelian Frobenius groups of type $(C_p \times C_p) \rtimes C_{p^2-1}$, in which there is precisely one conjugacy class of subgroups of order p ;

(II) the family of all metabelian Frobenius groups of type $(C_p \times C_p) \rtimes H$ in which there are precisely $p+1$ conjugacy classes of subgroups of order p .

We note that groups of type (I) are isomorphic to $\mathrm{AGL}_1(p^2)$, and any other Frobenius group of type $(C_p \times C_p) \rtimes C_m$ in which there is precisely one conjugacy class of subgroups of order p is a Frobenius subgroup of $\mathrm{AGL}_1(p^2)$. Furthermore, if $p = 2$, then the only group of type (I) is the alternating group $\mathfrak{A}_4$, while groups of type (II) only occur for odd prime numbers p. For this reason, we exclude the prime number 2 from all our calculations in this manuscript. We also emphasise that contrary to [BFL22, FL23], it is not possible to use block theoretical arguments in the present cases, because such groups possess only one p-block.

The paper is structured as follows. In Section 2 we introduce our notation and conventions. In Section 3, first we review some properties of Frobenius groups and their ordinary and Brauer characters. Then, we characterise metabelian Frobenius groups with elementary abelian Frobenius kernel. The trivial source character tables are calculated in Section 4 for groups of type (I), respectively in Section 5 for groups of type (II).

2. Preliminaries

2.1. **General notation.** Throughout, unless otherwise stated, we adopt the notation and conventions below. We let p denote an odd prime number and G denote a finite group of order divisible by p. We let $(K, \mathcal{O}, k)$ be a p-modular system, which we assume to be large enough for G and its subgroups. In other words, $\mathcal{O}$ denotes a complete discrete valuation ring of characteristic zero with field of fractions $K = \mathrm{Frac}(\mathcal{O})$ and residue field $\mathcal{O}$ is $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p, which we assume to be algebraically closed. For $R \in \{\mathcal{O}, k\}$, RG-modules are assumed to be finitely generated left RG-lattices, that is, free as R-modules, and we let R denote the trivial RG-lattice.

Given a positive integer n, we let C_n denote the cyclic group of order n. We let $\mathbf{O}_p(G)$ denote the largest normal p-subgroup of G, $\mathrm{Syl}_p(G)$ denote the set of all Sylow p-subgroups of G, $ccls(G)$ denote a set of representatives for the conjugacy classes of G, $[G]_{p'}$ denote a set of representatives for the p-regular conjugacy classes of G, and we let $G_{p'} := \{g \in G \mid p \nmid o(g)\}$. We recall that a group G with a normal subgroup N and a subgroup H is said to be the *internal semi-direct product of N by H*, written $G = N \rtimes H$, provided $G = NH$ and $N \cap H = \{1\}$.

Given $H \leq G$, an ordinary character ψ of H and χ an ordinary character of G, we write $\mathrm{Ind}_H^G(\psi)$ for the induction of ψ from H to G, $\mathrm{Res}_H^G(\chi)$ for the restriction of χ from G to H, $\chi^\circ := \chi|_{G_{p'}}$ for the reduction modulo p of χ, and 1_H for the trivial character of H. Given $N \trianglelefteq G$ and an ordinary character ν of G/N, we write $\mathrm{Inf}_{G/N}^G(\nu)$ for the inflation of χ from G/N to G. Similarly, we write $\mathrm{Ind}_H^G(L)$ for the induction of the kH-module L from H to G, $\mathrm{Res}_H^G(M)$

for the restriction of the kG-module M from G to H, and $\mathrm{Inf}_{G/N}^{G}(U)$ for the inflation of the $k[G/N]$-module U from G/N to G. Moreover, if M is a kG-module, then we denote by φ_M the Brauer character afforded by M, and if $Q \leq G$ then the Brauer quotient (or Brauer construction) of M at Q is the k-vector space $M[Q] := M^Q / \sum_{R<Q} \mathrm{tr}_R^Q(M^R)$, where M^Q denotes the fixed points of M under Q and tr_R^Q denotes the relative trace map. This vector space has a natural structure of a $kN_G(Q)$-module, but also of a $kN_G(Q)/Q$-module, and is equal to zero if Q is not a p-subgroup. Moreover, we use the abbreviation PIM to mean a *projective indecomposable module*. We assume that the reader is familiar with elementary notions of ordinary and modular representation theory of finite groups. We refer to [Lin18a, Web16, NT89, Hup98, CR90] for further standard notation and background results.

2.2. **Character tables and decomposition matrices.** We let $\mathrm{Irr}(G)$, $\mathrm{Lin}(G)$, and $\mathrm{IBr}_p(G)$ denote the set of all irreducible K-characters of G, the set of linear characters of G, and the set of all irreducible p-Brauer characters of G, respectively. We let

$$X(G) := \Big(\chi(g)\Big)_{\substack{\chi\in\mathrm{Irr}(G)\\ g\in ccls(G)}} \in K^{|\mathrm{Irr}(G)|\times|ccls(G)|}$$

denote the ordinary character table of G and we let

$$X(G,p') := \Big(\chi(g)\Big)_{\substack{\chi\in\mathrm{Irr}(G)\\ g\in[G]_{p'}}} \in K^{|\mathrm{Irr}(G)|\times|[G]_{p'}|}$$

denote the matrix obtained from $X(G)$ by removing the columns labelled by p-singular conjugacy classes. We recall that for any $\chi \in \mathrm{Irr}(G)$ there exist uniquely determined non-negative integers $d_{\chi\varphi}$ such that $\chi^\circ = \sum_{\varphi\in\mathrm{IBr}_p(G)} d_{\chi\varphi}\varphi$. Then, for any $\varphi \in \mathrm{IBr}_p(G)$, the *projective indecomposable character* associated to φ is

(1) $$\Phi_\varphi := \sum_{\chi\in\mathrm{Irr}(G)} d_{\chi\varphi}\chi\,.$$

The *p-decomposition matrix* of G is then

$$\mathrm{Dec}_p(G) := \Big(d_{\chi,\varphi}\Big)_{\substack{\chi\in\mathrm{Irr}(G)\\ \varphi\in\mathrm{IBr}_p(G)}} \in K^{|\mathrm{Irr}(G)|\times|\mathrm{IBr}_p(G)|}$$

and the *p-projective table* of G is

$$\Phi_p(G) := \Big(\Phi_\varphi(x)\Big)_{\substack{\varphi\in\mathrm{IBr}_p(G)\\ x\in[G]_{p'}}} \in K^{|\mathrm{IBr}_p(G)|\times[G]_{p'}}\,,$$

which is the table of Brauer character values of the projective indecomposable kG-modules. It follows from the definitions that

(2) $$\Phi_p(G) = \mathrm{Dec}_p(G)^t \cdot X(G,p')\,.$$

Finally, the character table of finite cyclic groups will play an essential role in our calculations, hence in this case we fix the following labelling of the irreducible characters and conjugacy classes.

Notation 2.1. If $G := \langle x \mid x^m = 1\rangle \cong C_m$ is a cyclic group of order $m \geq 1$, then we let $\zeta \in K$ denote a primitive m-th root of unity and we write the set of ordinary irreducible characters of G as $\mathrm{Irr}(G) = \{\xi_1, \ldots, \xi_m\}$, where

$$\xi_i(x^j) := \zeta^{(i-1)j}$$

for each $1 \leq i \leq m$ and each $0 \leq j \leq m-1$. This yields

$$X(C_m) := \Big(\xi_i(x^{j-1})\Big)_{\substack{1\leq i\leq m\\ 1\leq j\leq m}} = \Big(\zeta^{(i-1)(j-1)}\Big)_{\substack{1\leq i\leq m\\ 1\leq j\leq m}}\,.$$

2.3. **Trivial source character tables.** Given $R \in \{\mathcal{O}, k\}$, an RG-lattice M is called a *trivial source* RG-lattice if it is isomorphic to an indecomposable direct summand of an induced lattice $\operatorname{Ind}_Q^G(R)$, where $Q \leq G$ is a p-subgroup. In addition, if Q is of minimal order subject to this property, then Q is a vertex of M. It is clear, that up to isomorphism, there are only finitely many trivial source RG-lattices.

It is well-known that any trivial source kG-module M lifts in a unique way to a trivial source $\mathcal{O}G$-lattice $\widehat{M}$ (see e.g. [Ben98, Corollary 3.11.4]) and we denote by $\chi_{\widehat{M}}$ the K-character afforded by $\widehat{M}$. If M is a PIM, then $\chi_{\widehat{M}} = \Phi_\varphi$, wehere φ is the Brauer character afforded by the unique socle constituent of M.

We will study trivial source modules vertex by vertex. Hence, we denote by $\mathrm{TS}(G;Q)$ the set of isomorphism classes of indecomposable trivial source kG-modules with vertex Q. We notice that $\mathrm{TS}(G;\{1\})$ is precisely the set of isomorphism classes of PIMs of kG and if M is a PIM of kG, then $\chi_{\widehat{M}} = \Phi_\varphi$ where φ is the Brauer character afforded by the unique simple kG-module in the socle of M.

A p-subgroup $Q \leq G$ is a vertex of a trivial source kG-module M if and only if $M[Q]$ is a non-zero projective $k\overline{N}_G(Q)$-module. Moreover, if this is the case, then the $kN_G(Q)$-Green correspondent $f(M)$ of M is $M[Q]$ (viewed as a $kN_G(Q)$-module). Thus, there are bijections

$$\begin{array}{ccccc} \mathrm{TS}(G;Q) & \xrightarrow{\sim} & \mathrm{TS}(N_G(Q);Q) & \xrightarrow{\sim} & \mathrm{TS}(\overline{N}_G(Q);\{1\}) \\ M & \mapsto & f(M) & \mapsto & M[Q] \end{array}$$

where the inverse of the second map is given by the inflation from $\overline{N}_G(Q) := N_G(Q)/Q$ to $N_G(Q)$. These sets are also in bijection with the set of p'-conjugacy classes of $\overline{N}_G(Q)$.

Next, we let $a(kG, \mathrm{Triv})$ denote the *trivial source ring* of kG, which is defined to be the subring of the Grothendieck ring of kG generated by the set of all isomorphism classes of indecomposable trivial source kG-modules. By definition, the *trivial source character table of the group G at the prime p*, denoted $\mathrm{Triv}_p(G)$, is the species table of the trivial source ring of kG. See e.g. [BP84]. However, we follow [LP10, Section 4.10] and consider $\mathrm{Triv}_p(G)$ as the block square matrix defined according to the following convention.

Convention 2.2. First, fix a set of representatives $Q_1, \ldots, Q_r$ ($r \in \mathbb{Z}_{\geq 1}$) for the conjugacy classes of p-subgroups of G where $Q_1 := \{1\}$ and $Q_r \in \mathrm{Syl}_p(G)$. For each $1 \leq v \leq r$ set $\overline{N}_G(Q_v) := N_G(Q_v)/Q_v$. For each pair (Q_v, s) with $1 \leq v \leq r$ and $s \in [\overline{N}_G(Q_v)]_{p'}$ there is a ring homomorphism

$$\begin{array}{cccc} \tau^G_{Q_v,s}: & a(kG, \mathrm{Triv}) & \longrightarrow & K \\ & [M] & \mapsto & \varphi_{M[Q_v]}(s) \end{array}$$

mapping the class of a trivial source kG-module M to the value at s of the Brauer character $\varphi_{M[Q_v]}$ of the Brauer quotient $M[Q_v]$. (Note that the group G acts by conjugation on the pairs (Q_v, s) and the values of $\tau^G_{Q_v,s}$ do not depend on the choice of (Q_v, s) in its G orbit.) Then, for each $1 \leq i, v \leq r$ define a matrix

$$T_{i,v} := \left(\tau^G_{Q_v,s}([M])\right)_{\substack{M \in \mathrm{TS}(G;Q_i) \\ s \in [\overline{N}_G(Q_v)]_{p'}}}.$$

The *trivial source character table of G at the prime p* is then the block matrix

$$\mathrm{Triv}_p(G) := \Big[T_{i,v}\Big]_{\substack{1 \leq i \leq r \\ 1 \leq v \leq r}}.$$

Moreover, we label the rows of $\mathrm{Triv}_p(G)$ with the ordinary characters $\chi_{\widehat{M}}$ instead of the isomorphism classes of trivial source kG-modules M themselves.

In order to calculate the entries of $\mathrm{Triv}_p(G)$, we will use the following two well-known lemmata, the first of which describes the effect of the Brauer construction on characters of trivial source modules.

Lemma 2.3. *Let M be an indecomposable kG-module with a trivial source. Let $g \in G$ and write $g = g_p \cdot g_{p'}$ with g_p a p-element and $g_{p'}$ a p'-element. Then*

$$\chi_{\widehat{M}}(g) = \tau^G_{\langle g_p\rangle, g_{p'}}([M])\,.$$

Proof. First note that since g_p and $g_{p'}$ commute, certainly $g_{p'} \in C_G(g_p)$. Next, let $L := M[\langle g_p\rangle]$ (seen as a $kN_G(\langle g_p\rangle)$-module) and $M_{g_p} := \mathrm{Res}^{N_G(\langle g_p\rangle)}_{C_G(g_p)}(L)$. Thus, it follows from [Ric96, Lemma 6.2] (or [BP84, 10.13. LEMMA]) that

$$\chi_{\widehat{M}}(g) = \chi_{\widehat{M}}(g_p g_{p'}) = \chi_{\widehat{M_{g_p}}}(g_{p'})$$

and on the other hand, it follows from the definitions that

$$\tau^G_{\langle g_p\rangle, g_{p'}}([M]) = \chi_{\widehat{L}}(g_{p'}) = (\mathrm{Res}^{N_G(\langle g_p\rangle)}_{C_G(g_p)}(\chi_{\widehat{L}}))(g_{p'}) = \chi_{\widehat{M_{g_p}}}(g_{p'})\,,$$

proving the claim. □

The second lemma lets us describe certain blocks of the trivial source character table using ordinary and Brauer characters.

Lemma 2.4. *Let $\mathrm{Triv}_p(G) = [T_{i,v}]_{1\le i,v\le r}$ be the trivial source character table of the finite group G at p. Then, the following assertions hold:*

(a) *$T_{i,v} = \mathbf{0}$ if $Q_v \not\le_G Q_i$, so in particular $T_{i,v} = \mathbf{0}$ for every $1 \le i < v \le r$;*

(b) *$T_{i,i} = \Phi_p(\overline{N}_G(Q_i)) = \mathrm{Dec}_p(\overline{N}_G(Q_i))^t \cdot X(\overline{N}_G(Q_i), p')$ for every $1 \le i \le r$;*

(c) *$T_{i,1} = \big(\chi_{\widehat{M}}(s)\big)_{M\in \mathrm{TS}(G;Q_i), s\in [G]_{p'}}$ for every $1 \le i \le r$.*

Proof. Assertion (a) is given by [LP10, Lemma 4.10.11(b)]. The first equality in assertion (b) is given by [LP10, Lemma 4.10.11(c)] and the second equality follows from Equation (2) above. Now, if $v = 1$ and $1 \le i \le r$, then $M[Q_v] = M[\{1\}] = M$, so $\tau^G_{\{1\},s}([M]) = \varphi_M(s) = \chi_{\widehat{M}}(s)$ for every $M \in \mathrm{TS}(G;Q_i)$ and every $s \in [G]_{p'}$, proving assertion (c). □

We refer the reader to the survey [Las23] and to our previous paper [BFL22, §2] for further details and further properties of trivial source modules and trivial source character tables. However, we mention the following result from the thesis of the first author, which will be crucial.

Proposition 2.5 ([Böh24, Proposition 3.1.15])**.** *Assume G is a finite group with a normal Sylow p-subgroup $P \trianglelefteq G$ such that G/P is abelian. Let Q be a p-subgroup of G. Then, we have $P \cap N_G(Q) = \mathbf{O}_p(N_G(Q)) \in \mathrm{Syl}_p(N_G(Q))$ and by the Schur–Zassenhaus Theorem, we may choose a complement C of $P \cap N_G(Q)$ in $N_G(Q)$. Let S be a simple kC-module, viewed as a simple $k[QC/C]$-module via the canonical isomorphism $QC/Q \cong C$. Set*

$$L := \mathrm{Ind}^{\overline{N}_G(Q)}_{QC/Q}(S) \qquad \text{and} \qquad U := \mathrm{Inf}^{N_G(Q)}_{\overline{N}_G(Q)}(L)\,.$$

Then, the following assertions hold:

(a) *L is a projective indecomposable $k\overline{N}_G(Q)$-module; and*

(b) *$M := \mathrm{Ind}^G_{N_G(Q)}(U)$ is indecomposable, hence a trivial source kG-module with vertex Q.*

In particular, any element of $\mathrm{TS}(G;Q)$ can be obtained in this way.

3. Background material on Frobenius groups

We start by reviewing basic definitions and results about the character theory of Frobenius groups.

3.1. **Frobenius groups.** Recall that a finite group G admitting a non-trivial proper subgroup H such that

$$H \cap gHg^{-1} = \{1\}$$

for each $g \in G \backslash H$ is called a *Frobenius group* with *Frobenius complement* H (or a *Frobenius group with respect to* H). Frobenius proved that in such a group there exists a uniquely determined normal subgroup F such that G is the internal semi-direct product of F by H (i.e. $G = FH$ and $F \cap H = \{1\}$); concretely,

$$F = \{1\} \cup \left(G \setminus \bigcup_{g \in G} gHg^{-1}\right).$$

The normal subgroup F is called the *Frobenius kernel* of G. See e.g. [CR90, §14A]. In the sequel, we write Frobenius groups with respect to H as $F \rtimes H$.

We will use the following well-known properties of Frobenius groups.

Lemma 3.1. *Let G be a Frobenius group with Frobenius complement H and Frobenius kernel F. Then the following assertions hold.*

(a) *If H is abelian, then H is cyclic.*

(b) *The integer $|H|$ divides $|F| - 1$. In particular $|G : F|$ and $|F|$ are coprime integers, hence F is characteristic in G.*

(c) *For each $f \in F \setminus \{1\}$ we have $C_G(f) \leq F$.*

Proof. Assertion (a) is due to Burnside and follows directly from [Hup98, Lemma 16.7b)]. Assertion (b) is given by [Hup98, 16.6a) Lemma]. Assertion (c) is given by [CR90, (14.4) Proposition (i)]. □

3.2. **Characters of Frobenius groups.** The ordinary characters of Frobenius groups are well-known and given by the following theorem.

Theorem 3.2 ([CR90, (14.4) Proposition])**.** *Let G be a Frobenius group with Frobenius complement H and Frobenius kernel F. Then*

$$\mathrm{Irr}(G) = \{\mathrm{Inf}_{G/F}^{G}(\psi) \mid \psi \in \mathrm{Irr}(G/F)\} \sqcup \{\mathrm{Ind}_{F}^{G}(\nu) \mid \nu \in T\},$$

where T is a set of representatives for the orbits of the action of G by conjugation on $\mathrm{Irr}(F) \backslash \{1_F\}$.

Notice that the first set consists of the irreducible characters of G which contain F in their kernels, whereas the second set consists of those irreducible characters of G which do not contain F in their kernels.

Lemma 3.3 ([Hup98, 18.7 Theorem (b)])**.** *Let G be a Frobenius group with Frobenius complement H and Frobenius kernel F. Let ρ_H denote the regular character of H. If $\nu \in \mathrm{Irr}(F) \setminus \{1_F\}$, then*

$$\mathrm{Res}_{H}^{G}(\mathrm{Ind}_{F}^{G}(\nu)) = \nu(1) \cdot \rho_H .$$

Proposition 3.4. *Let G be a Frobenius group with cyclic Frobenius complement $H \cong C_m$ for some integer $m \geq 2$ and abelian Frobenius kernel F of order p^r for some positive integer $r \geq 1$. Then the following assertions hold:*

(a) $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_{m+\frac{p^r-1}{m}}\}$ *where for each $1 \leq i \leq m$*

$$\chi_i := \mathrm{Inf}_{G/F}^{G}(\xi_i)$$

with $\xi_i \in \mathrm{Irr}(C_m)$ as defined in Notation 2.1, and

$$\{\chi_{m+1}, \dots, \chi_{m+\frac{p^r-1}{m}}\} = \{\mathrm{Ind}_F^G(\nu) \mid \nu \in T\}$$

where T is a set of representatives for conjugation action of G on $\mathrm{Irr}(F) \setminus \{1_F\}$;

(b) $\mathrm{IBr}_p(G) = \{\varphi_1, \dots, \varphi_m\}$ *where $\varphi_i := \chi_i^\circ$ for each $1 \leq i \leq m$;*

(c) $\mathrm{Dec}_p(G)$ *is as given in Table 1.*

	φ_1	φ_2	$\cdots\cdots$	φ_{m-1}	φ_m
χ_1	1	0	$\cdots\cdots$	0	0
χ_2	0	1	$\ddots$	$\vdots$	$\vdots$
χ_3	0	0	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	1	0
χ_m	0	0	$\cdots\cdots$	0	1
χ_{m+1}	1	1	$\cdots\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\chi_{m+\frac{p^r-1}{m}}$	1	1	$\cdots\cdots$	1	1

TABLE 1

Proof. Recall from Lemma 3.1 that $\gcd(m, p) = 1$.

(a) First, it is clear from Theorem 3.2 that G has m pairwise distinct ordinary irreducible characters which are inflated from $G/F \cong H \cong C_m$ to G. Moreover,

$$\begin{aligned}|\,\mathrm{Irr}(G)| &= |\,\mathrm{Irr}(G/F)| + |\{\mathrm{Ind}_F^G(\nu) \mid \nu \in T\}| \\ &= m + \frac{|F|-1}{|H|} = m + \frac{p^r-1}{m}\end{aligned}$$

where the last-but-one equality holds by [Hup98, 18.7 Theorem (b)].

(b) It is well-known that $|\,\mathrm{IBr}_p(G)|$ is equal to the number of p'-conjugacy classes of G. As G is a semi-direct product of the normal p-subgroup F by the abelian p'-subgroup H, clearly $ccls(H) = H$ is a set of representatives of p'-conjugacy classes of G, proving that $|\,\mathrm{IBr}_p(G)| = |H| = m$. Now, as the reductions modulo p of the m linear characters $\chi_1, \dots, \chi_m$ of G are pairwise distinct linear Brauer characters, they already account for all the irreducible Brauer characters of G. The claim follows.

(c) It is immediate from part (b) that the first m rows of $\mathrm{Dec}_p(G)$ are given by the identity matrix of size $m \times m$. Let now $\chi_j \in \mathrm{Irr}(G)$ with $m+1 \leq j \leq m + \frac{p^r-1}{m}$. By (a), there exists a character $\nu \in \mathrm{Irr}(F) \setminus \{1_F\}$ such that $\chi_j = \mathrm{Ind}_F^G(\nu)$. As $G_{p'} = H$ and is abelian, we have

$$\chi_j^\circ = (\mathrm{Ind}_F^G(\nu))^\circ = \mathrm{Res}_H^G(\mathrm{Ind}_F^G(\nu)) = \rho_H = \sum_{\xi \in \mathrm{Irr}(H)} \xi = \sum_{i=1}^{m} \varphi_i,$$

where the third equality follows from Lemma 3.3 and the last equality follows from (a) and (b).

□

3.3. **Frobenius groups of type $(C_p \times C_p) \rtimes H$.** In this article, the aim is to focus on Frobenius groups G with cyclic Frobenius complement of order m and elementary abelian Frobenius kernel of order p^2 where p is an odd prime number. In particular, we will compute the trivial source character tables $\mathrm{Triv}_p(G)$ in the following two extremal cases: first the case, in which there is precisely one G-conjugacy class of cyclic subgroups of order p (we will call this the *maximal fusion case*); second, the case in which there are precisely $p+1$ G-conjugacy classes of cyclic subgroups of order p (we call this the *minimal fusion case*). In this subsection, we characterise such groups.

Given integers $m, n > 1$, we denote by $\mathrm{MetaFrob}(m)$ the set of isomorphism classes of metabelian Frobenius groups with Frobenius complement of order m and we set

$$\mathrm{MetaFrob}(m, n) := \{G \ \in \mathrm{MetaFrob}(m) \ \mid \ |G| = mn\}.$$

Note that Frobenius groups G with cyclic Frobenius complement of order m and elementary abelian Frobenius kernel of order p^2 comprise all elements of $\mathrm{MetaFrob}(m, p^2)$ whose Frobenius kernels are not cyclic.

It is known that the cardinality of $\mathrm{MetaFrob}(m, n)$ can be expressed in terms of the group of units $(\mathbb{Z}/m\mathbb{Z})^\times$, Euler's totient function φ, and the prime factorization

$$n = p_1^{a_1} \cdot \ldots \cdot p_r^{a_r},$$

where the positive integers $p_1, \ldots, p_r$ are assumed to be pairwise distinct prime numbers. For convenience, we identify $(\mathbb{Z}/m\mathbb{Z})^\times$ with

$$\mathcal{E} := \{i \in \mathbb{Z} \mid 0 < i < m \text{ and } \gcd(m, i) = 1\}.$$

Moreover, for integers a and b coprime to m let $d(a, b)$ denote the order of the subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$ generated by $a + m\mathbb{Z}$ and $b + m\mathbb{Z}$. Also, if $0 < v \in \mathbb{Z}$, let $P(u, v) = 0$ if u is not a nonnegative integer and let $P(u, v)$ denote the coefficient of x^u in the power series expansion of $\prod_{i=1}^{\infty} (1 - x^i)^{-v}$ otherwise.

Proposition 3.5 ([BH98, Theorem 11.7.]). *Let $m, n > 1$ be integers and let $n = p_1^{a_1} \cdot \ldots \cdot p_r^{a_r}$ be the prime factorisation of n. Then,*

$$|\mathrm{MetaFrob}(m, n)| = \frac{1}{\varphi(m)} \sum_{e \in \mathcal{E}} \prod_{i=1}^{r} P\left(\frac{a_i}{d(e, p_i)}, \frac{\varphi(m)}{d(e, p_i)}\right).$$

Corollary 3.6. *Let p be an odd prime number and let $m > 1$ be an integer such that $m \nmid (p-1)$. Then the following assertions hold:*

(a) $|\mathrm{MetaFrob}(m, p^2)| = 1$*;*

(b) *if* $m = p^2 - 1$ *then the unique element of* $\mathrm{MetaFrob}(p^2 - 1, p^2)$ *is the affine linear group* $\mathrm{AGL}_1(p^2)$ *and can be identified with the subgroup*

$$\mathcal{G} := \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_{p^2}) \mid a \in \mathbb{F}_{p^2}^\times, b \in \mathbb{F}_{p^2} \right\}$$

of $\mathrm{GL}_2(\mathbb{F}_{p^2})$*, which is a Frobenius group with Frobenius kernel and Frobenius complement*

$$\mathcal{F} := \left\{ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_{p^2}) \mid b \in \mathbb{F}_{p^2} \right\} \quad \text{and} \quad \mathcal{H} := \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_{p^2}) \mid a \in \mathbb{F}_{p^2}^\times \right\}$$

respectively;

(c) *if* $p < m < p^2 - 1$ *then the unique element of* $\mathrm{MetaFrob}(m, p^2)$ *can be identified with the subgroup* $\widetilde{\mathcal{G}} = \mathcal{F}\widetilde{\mathcal{H}} = \mathcal{F} \rtimes \widetilde{\mathcal{H}}$ *of* $\mathcal{G}$*, where* $|\widetilde{\mathcal{H}}| = m$*.*

Proof. (a) Using the notation of Proposition 3.5 we have $n = p^2$, hence $r = 1$, $a_1 = 2$, and $p_1 = p$. Thus, it follows from Proposition 3.5 that

$$|\mathrm{MetaFrob}(m, p^2)| = \frac{1}{\varphi(m)} \sum_{e \in \mathcal{E}} P\left(\frac{2}{d(e,p)}, \frac{\varphi(m)}{d(e,p)}\right).$$

By the definition of the function P, the above sum does not vanish if and only if $d(e,p) \in \{1, 2\}$. As the order of $p + m\mathbb{Z}$ in $(\mathbb{Z}/m\mathbb{Z})^\times$ is two (since $m \mid p^2 - 1$ but $m \nmid p - 1$), the case $d(e,p) = 1$ does not occur.

Now, we claim that $d(e,p) = 2$ if and only if $e \equiv p \pmod{m}$ or $e \equiv 1 \pmod{m}$. By Lemma 3.1(b) we have $m \mid p^2 - 1$, so $p^2 \equiv 1 \pmod{m}$. Hence, if $e \equiv p \pmod{m}$ or $e \equiv 1 \pmod{m}$ then $d(e,p) = 2$ because $\langle p + m\mathbb{Z} \rangle = \{p + m\mathbb{Z}, p^2 + m\mathbb{Z}\}$. Conversely, if $d(e,p) = 2$ then $|\langle p + m\mathbb{Z} \rangle| = 2$. Hence, $e + m\mathbb{Z}$ cannot contribute to any new element of $\langle p + m\mathbb{Z} \rangle$ and it follows that $e \equiv p \pmod{m}$ or $e \equiv 1 \pmod{m}$.
Therefore, we have:

$$\begin{aligned} |\mathrm{MetaFrob}(m, p^2)| &= \frac{1}{\varphi(m)} \cdot \left(P\left(\frac{2}{2}, \frac{\varphi(m)}{2}\right) + P\left(\frac{2}{2}, \frac{\varphi(m)}{2}\right) \right) \\ &= \frac{2}{\varphi(m)} \cdot P\left(1, \frac{\varphi(m)}{2}\right). \end{aligned}$$

Finally, by [BH98, Remark 11.13.(C)], we obtain $P\left(1, \frac{\varphi(m)}{2}\right) = \frac{\varphi(m)}{2}$, proving our assertion.

(b) Assertion (b) is well-known and follows for example from [Jac12, Exercise 5.15.7].

(c) The group $\widetilde{\mathcal{G}} = \mathcal{F} \rtimes \widetilde{\mathcal{H}}$ is obviously a subgroup of $\mathcal{G}$. Moreover, as $\widetilde{\mathcal{H}} < \mathcal{H}$ and

$$\widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{H}} = \{f \cdot \widetilde{h} \in \widetilde{\mathcal{G}} \mid f \in \mathcal{F} \setminus \{1\}, \widetilde{h} \in \widetilde{\mathcal{H}}\} \quad \subset \quad \mathcal{G} \setminus \mathcal{H} = \{f \cdot h \in \mathcal{G} \mid f \in \mathcal{F} \setminus \{1\}, h \in \mathcal{H}\},$$

it follows from the definition that $\mathcal{F} \rtimes \widetilde{\mathcal{H}}$ is again a Frobenius group.

□

Proposition 3.7. *Let G be a Frobenius group with Frobenius complement $H \cong C_m$ and Frobenius kernel $F \cong C_p \times C_p$.*

(a) *The number of G-conjugacy classes of subgroups of G of order p equals 1 if and only if $m = (p + 1) \cdot \gcd(p - 1, m)$.*

(b) *If the number of G-conjugacy classes of subgroups of G of order p equals $p+1$, then H is cyclic of order dividing $p-1$.*

Proof. Write $F = \langle x \rangle \times \langle y \rangle$ and $H = \langle h \rangle$. As F is elementary abelian of order p^2, there are precisely $p+1$ subgroups of F of order p, namely

$$R_i := \langle x \cdot y^i \rangle \quad (0 \leq i \leq p-1) \quad \text{and} \quad R_p := \langle y \rangle,$$

and we let $X := \{R_0, \ldots, R_p\}$.

(a) First, suppose there is a unique G-conjugacy class of subgroups of G of order p. The group G acts on the set X by conjugation. We deduce from the orbit-stabiliser theorem that, for all $C \in X$,

$$p+1 = |\mathrm{Orb}(C)| = |G : N_G(C)| = \frac{|G|}{|N_G(C)|},$$

where $\mathrm{Orb}(C)$ denotes the orbit of C under the action of G. Hence,

$$(p+1) \cdot |N_G(C)| = |G| = p^2 \cdot m.$$

Moreover, we know from Lemma 3.1(b) that $m \mid (p+1) \cdot (p-1)$ (hence $\gcd(m,p) = 1$), and $F \leq N_G(C)$, so $p^2 \mid |N_G(C)|$. It follows that $(p+1) \mid m$, so $m = (p+1) \cdot \gcd(p-1, m)$ and $|N_G(C)| = p^2 \cdot \gcd(p-1, m)$.

Conversely, assume that $m = (p+1) \cdot \gcd(m, p-1)$. Let $C \in X$. Since $F \leq N_G(C)$, as above, we can write $|N_G(C)| = p^2 \cdot b$ where b is a positive integer such that $b \mid m$. First, we claim that $b \mid (p-1)$. Indeed, $N_G(C)$ acting by conjugation on C, we may consider the induced group homomorphism

$$\Theta : N_G(C) \longrightarrow \mathrm{Aut}(C) \cong C_{p-1}, g \mapsto c_g$$

where $c_g : C \longrightarrow C, c \mapsto gcg^{-1}$ is the automorphism of conjugation by g. Since C is a subgroup of the Frobenius kernel F, for any $c \in C \setminus \{1\}$ we have $C_G(c) \leq F$ by Lemma 3.1(c). Thus, $\ker(\Theta) = F$ and

$$b = \frac{|N_G(C)|}{|F|} \,\Big|\, |\mathrm{Aut}(C)| = p-1.$$

Therefore, the orbit-stabiliser theorem yields

$$|\mathrm{Orb}(C)| \cdot |N_G(C)| = |G| = p^2 \cdot m$$

that is,

$$|\mathrm{Orb}(C)| \cdot b = m = (p+1) \cdot \gcd(m, p-1)\,.$$

We deduce that $(p+1) \mid |\mathrm{Orb}(C)|$, so the only possibility is $|\mathrm{Orb}(C)| = p+1$, proving our claim.

(b) Let $C \in X$. Since the number of G-conjugacy classes of subgroups of G of order p equals $p+1$, we have $|\mathrm{Orb}(C)| = 1$ and $N_G(C) = G$. As in part (a), the kernel of the homomorphism

$$\Theta : N_G(C) \longrightarrow \mathrm{Aut}(C), g \mapsto c_g$$

is F, implying that

$$m = \frac{|G|}{|F|} \,\Big|\, |\mathrm{Aut}(C)| = p-1\,.$$

□

Remark 3.8. Notice that for the prime number 2 there is, up to isomorphism, only one Frobenius group of type $(C_2 \times C_2) \rtimes C_m$, namely the alternating group $\mathfrak{A}_4 = V_4 \rtimes \langle(1\ 2\ 3)\rangle$, where V_4 is the Klein-four group. Indeed, as $m \mid (p^2 - 1) = 3$, the only possibility is $m = 3$, and the only other non-abelian group of order 12 is the dihedral group of order 12, which does not have any normal subgroup isomorphic to $C_2 \times C_2$. The trivial source character table $\mathrm{Triv}_2(\mathfrak{A}_4)$ can be found for example in [Böh24], or in [BFL22] through the isomorphism $\mathfrak{A}_4 \cong \mathrm{PSL}_2(3)$.

4. The maximal fusion case

We now turn to the computation of the trivial source character tables of metabelian Frobenius group with Frobenius kernel $F \cong C_p \times C_p$ and cyclic Frobenius complement $H \cong C_{p^2-1}$. Recall from Corollary 3.6 that, up to isomorphism, there is only one group of this type, namely $\mathrm{AGL}_1(p^2)$ and, in this case, precisely one conjugacy class of subgroups of order p by Proposition 3.7(a). The aforementioned corollary and proposition also tell us that any other Frobenius group of type $(C_p \times C_p) \rtimes C_m$ with precisely one conjugacy class of subgroups of order p is then a subgroup of $\mathrm{AGL}_1(p^2)$. For this reason, below, we only calculate $\mathrm{Triv}_p(\mathrm{AGL}_1(p^2))$.

Notation 4.1. Throughout this section, we assume that

$$G = \mathcal{G} = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_{p^2}) \mid a \in \mathbb{F}_{p^2}^{\times}, b \in \mathbb{F}_{p^2} \right\} = \mathcal{F} \rtimes \mathcal{H}$$

with

$$\mathcal{F} := \left\{ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_{p^2}) \mid b \in \mathbb{F}_{p^2} \right\} \quad \text{and} \quad \mathcal{H} := \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_{p^2}) \mid a \in \mathbb{F}_{p^2}^{\times} \right\}$$

as in Corollary 3.6. We choose a generator

$$h := \begin{pmatrix} a' & 0 \\ 0 & 1 \end{pmatrix}$$

of $\mathcal{H}$ where a' is a generator of $\mathbb{F}_{p^2}^{\times}$. We let $x, y \in \mathcal{F}$ be elements of order p such that $\mathcal{F} = \langle x \rangle \times \langle y \rangle$ where x is chosen such that

$$\langle x \rangle = \{ \begin{pmatrix} 1 & 0 \\ b' & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_{p^2}) \mid (b')^p = b' \}$$

We choose the following set of representatives of the conjugacy classes of G:

$$\{1, h, h^2, \dots, h^{p^2-2}, x\}.$$

By Proposition 3.7, we can choose the following set of representatives for the G-conjugacy classes of p-subgroups of G:

$$\begin{aligned} Q_1 &:= \{1\}, \\ Q_2 &:= \langle x \rangle, \\ Q_3 &:= \mathcal{F}. \end{aligned}$$

As in Proposition 3.4, we let $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_{p^2}\}$ where for each $1 \le i \le p^2 - 1$ we set $\chi_i := \mathrm{Inf}_{G/F}^{G}(\xi_i)$ with $\xi_i \in \mathrm{Irr}(\mathcal{H})$ as defined in Notation 2.1, and $\chi_{p^2} := \mathrm{Ind}_{\mathcal{F}}^{G}(\nu)$ for some $\nu \in \mathrm{Irr}(\mathcal{F}) \setminus \{1_{\mathcal{F}}\}$.

Lemma 4.2. *With notation of Notation 4.1, we have:*

(a) $N_G(Q_1) = G$ *and* $\overline{N}_G(Q_1) \cong G$*;*

(b) $N_G(Q_2) = \mathcal{F} \rtimes \langle h^{p+1} \rangle$ *and* $\overline{N}_G(Q_2) \cong \langle y \rangle \rtimes \langle h^{p+1} \rangle$*;*

(c) $N_G(Q_3) = G$ *and* $\overline{N}_G(Q_3) \cong \mathcal{H}$*.*

Proof. Assertions (a) and (c) are straightforward from the definitions. Assertion (b) was proved in part (a) of the proof of Proposition 3.7. □

Proposition 4.3. *Let* ζ *be a fixed primitive* (p^2-1)*-th root of unity in* K *and let* $\omega := \zeta^{(p+1)}$*. Then the following assertions hold.*

(a) *The ordinary character table of* G *is as given in Table 2.*

	1_G	$h^j \ (1 \le j \le p^2-2)$	x
$\chi_i \ (1 \le i \le p^2-1)$	1	$\zeta^{(i-1)j}$	1
χ_{p^2}	p^2-1	0	-1

TABLE 2. Ordinary character table of $\mathrm{AGL}_1(p^2)$.

(b) *The ordinary character table of* $\overline{N}_G(Q_2)$ *is as given in Table 3, where following Proposition 3.4, we let* $\mathrm{Irr}(\overline{N}_G(Q_2)) = \{\theta_1, \ldots, \theta_p\}$ *where for each* $1 \le i \le p-1$ *we set* $\theta_i := \mathrm{Inf}^{\overline{N}_G(Q_2)}_{\overline{N}_G(Q_2)/\langle y \rangle}(\xi_i)$ *with* $\xi_i \in \mathrm{Irr}(\langle h^{p+1} \rangle)$ *as defined in Notation 2.1, and* $\theta_p := \mathrm{Ind}^{\overline{N}_G(Q_2)}_{\langle y \rangle}(\nu)$ *for some* $\nu \in \mathrm{Irr}(\langle y \rangle) \setminus \{1_{\langle y \rangle}\}$ *.*

	1_G	$h^{j(p+1)} \ (1 \le j \le p-2)$	y
$\theta_i \ (1 \le i \le p-1)$	1	$\omega^{(i-1)j}$	1
θ_p	$p-1$	0	-1

TABLE 3. Ordinary character table of $\overline{N}_G(Q_2)$.

(c) *Setting* $\varphi_i := \chi_i^\circ$ *for each* $1 \le i \le p^2-1$*, then* $\mathrm{IBr}_p(G) = \{\varphi_1, \ldots, \varphi_{p^2-1}\}$ *and* $\mathrm{Dec}_p(G)$ *is as given in Table 4.*

(d) *Setting* $\psi_i := \theta_i^\circ$ *for each* $1 \le i \le p-1$*, then* $\mathrm{IBr}_p(\overline{N}_G(Q_2)) = \{\psi_1, \ldots, \psi_{p-1}\}$ *and* $\mathrm{Dec}_p(\overline{N}_G(Q_2))$ *is as given in Table 5.*

Proof. (a) For each $1 \le i \le p^2-1$, we have $\chi_i = \mathrm{Inf}^G_{G/F}(\xi_i)$ with $\xi_i \in \mathrm{Irr}(\mathcal{H})$. It follows from Notation 2.1 that $\chi_i(h^j) = \xi_i(h^j) = \zeta^{(i-1)j}$ for each $1 \le j \le p^2-2$. Moreover $\chi_i(x) = 1$, as $x \in \mathcal{F}$. Then, $\chi_{p^2}(1) = p^2-1$ as χ_{p^2} is induced from a linear character of $\mathcal{F}$ to G. Using the definition of an induced character (see e.g. [Lin18b, Definition 3.1.19]) we obtain $\chi_{p^2}(h^j) = 0$ for each $1 \le j \le p^2-2$. Finally, $\chi_{p^2}(x) = -1$ follows from the 1st orthogonality relations applied to χ_{p^2} and the trivial character.

(b) Analogous to (a).

(c) This is immediate from Proposition 3.4(c).

	φ_1	φ_2	$\cdots$	φ_{p^2-2}	φ_{p^2-1}
χ_1	1	0	$\cdots$	0	0
χ_2	0	1	$\ddots$	$\vdots$	$\vdots$
χ_3	0	0	$\ddots$		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	0	0
			$\ddots$	1	0
χ_{p^2-1}	0	0	$\cdots$	0	1
χ_{p^2}	1	1	$\cdots$	1	1

TABLE 4. p-decomposition matrix of $\mathrm{AGL}_1(p^2)$

	ψ_1	ψ_2	$\cdots$	ψ_{p-2}	ψ_{p-1}
θ_1	1	0	$\cdots$	0	0
θ_2	0	1	$\ddots$	$\vdots$	$\vdots$
θ_3	0	0	$\ddots$		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	0	0
			$\ddots$	1	0
θ_{p-1}	0	0	$\cdots$	0	1
θ_p	1	1	$\cdots$	1	1

TABLE 5. p-decomposition matrix of $\overline{N}_G(Q_2)$

(d) This is immediate from Proposition 3.4(c).

□

Theorem 4.4. *Assume $G = \mathrm{AGL}_1(p^2)$. Let ζ denote a primitive (p^2-1)-th root of unity in K. Then, the trivial source character table $\mathrm{Triv}_p(G)$ of G is given as follows:*

(a) $T_{1,2} = T_{1,3} = T_{2,3} = \mathbf{0}$ *;*

(b) *the matrices $T_{i,1}$ with $1 \leq i \leq 3$ are as given in Table 6;*

(c) *the matrices $T_{2,2}$ and $T_{3,2}$ are as given in Table 7;*

(d) *the matrix $T_{3,3}$ is as given in Table 8.*

		1_H	h	$\cdots$	$h^{b(p+1)-1}$ $(1 \leq b \leq p-2)$	$h^{b(p+1)}$ $(1 \leq b \leq p-2)$	$h^{b(p+1)+1}$ $(1 \leq b \leq p-2)$	$\cdots$	h^{p^2-2}
$T_{1,1}$	$\chi_1 + \chi_{p^2}$	p^2	1	$\cdots$	1	1	1	$\cdots$	1
	$\chi_2 + \chi_{p^2}$	p^2	ζ	$\cdots$	$\zeta^{b(p+1)-1}$	$\zeta^{b(p+1)}$	$\zeta^{b(p+1)+1}$	$\cdots$	ζ^{p^2-2}
	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
	$\chi_{p^2-1} + \chi_{p^2}$	p^2	ζ^{p^2-2}	$\cdots$	$\zeta^{(b(p+1)-1)\cdot(p^2-2)}$	$\zeta^{b(p+1)\cdot(p^2-2)}$	$\zeta^{(b(p+1)+1)\cdot(p^2-2)}$	$\cdots$	$\zeta^{(p^2-2)^2}$
$T_{2,1}$	$\chi_{p^2} + \sum_{a=0}^{p} \chi_{a(p-1)+1}$	$p(p+1)$	0	$\cdots$	0	$p+1$	0	$\cdots$	0
	$\chi_{p^2} + \sum_{a=0}^{p} \chi_{a(p-1)+2}$	$p(p+1)$	0	$\cdots$	0	$\zeta^{b(p+1)} \cdot (p+1)$	0	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
	$\chi_{p^2} + \sum_{a=0}^{p} \chi_{a(p-1)+(p-1)}$	$p(p+1)$	0	$\cdots$	0	$\zeta^{(p-2)\cdot b(p+1)} \cdot (p+1)$	0	$\cdots$	0
$T_{3,1}$	χ_1	1	1	$\cdots$	1	1	1	$\cdots$	1
	χ_2	1	ζ	$\cdots$	$\zeta^{b(p+1)-1}$	$\zeta^{b(p+1)}$	$\zeta^{b(p+1)+1}$	$\cdots$	ζ^{p^2-2}
	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
	χ_{p^2-1}	1	ζ^{p^2-2}	$\cdots$	$\zeta^{(b(p+1)-1)\cdot(p^2-2)}$	$\zeta^{b(p+1)\cdot(p^2-2)}$	$\zeta^{(b(p+1)+1)\cdot(p^2-2)}$	$\cdots$	$\zeta^{(p^2-2)^2}$

TABLE 6. $T_{i,1}$ for $1 \leq i \leq 3$.

		1_H	h^{p+1}	$h^{2\cdot(p+1)}$	$\cdots$	$h^{(p-2)\cdot(p+1)}$
$T_{2,2}$	$\chi_{p^2}+\sum_{a=0}^{p}\chi_{a(p-1)+1}$	p	1	1	$\cdots$	1
	$\chi_{p^2}+\sum_{a=0}^{p}\chi_{a(p-1)+2}$	p	ζ^{p+1}	$\zeta^{2\cdot(p+1)}$	$\cdots$	$\zeta^{(p-2)\cdot(p+1)}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
	$\chi_{p^2}+\sum_{a=0}^{p}\chi_{a(p-1)+p-1}$	p	$\zeta^{(p-2)\cdot(p+1)}$	$\zeta^{2\cdot(p-2)\cdot(p+1)}$	$\cdots$	$\zeta^{(p-2)^2\cdot(p+1)}$
$T_{3,2}$	χ_1	1	1	1	$\cdots$	1
	χ_2	1	$\zeta^{(p+1)}$	$\zeta^{2\cdot(p+1)}$	$\cdots$	$\zeta^{(p-2)\cdot(p+1)}$
	χ_3	1	$\zeta^{2\cdot(p+1)}$	$\zeta^{4\cdot(p+1)}$	$\cdots$	$\zeta^{2\cdot(p-2)\cdot(p+1)}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
	χ_{p^2-1}	1	$\zeta^{(p^2-2)\cdot(p+1)}$	$\zeta^{2\cdot(p^2-2)\cdot(p+1)}$	$\cdots$	$\zeta^{(p^2-2)\cdot(p-2)\cdot(p+1)}$

TABLE 7. $T_{i,2}$ for $2 \leq i \leq 3$.

		h^j $(0 \leq j \leq p^2-2)$
$T_{3,3}$	χ_i $(1 \leq i \leq p^2-1)$	$\zeta^{(i-1)j}$ $(1 \leq i \leq p^2-1)$

TABLE 8. $T_{3,3}$.

Proof. (a) The claim is immediate from Lemma 2.4(a).

(b) The matrix $T_{1,1}$. By Lemma 2.4(b), we have

$$T_{1,1} = \Phi_p(G) = \mathrm{Dec}_p(G)^t \cdot X(G,p').$$

Hence, the values of $T_{1,1}$ in Table 6 are obtained from Table 2 and Table 4. The labels of the rows of $T_{1,1}$ are the ordinary characters of the PIMs of kG and can be read off from the decomposition matrix in Table 4.

The matrix $T_{2,1}$. By Lemma 2.4(c), we have

$$T_{2,1} = \big(\chi_{\widehat{M}}(s)\big)_{M\in\mathrm{TS}(G;Q_2),s\in[G]_{p'}}.$$

Moreover, we sort the modules in $\mathrm{TS}(G;Q_2)$ according to the labelling of the rows of $\Phi_p(\overline{N}_G(Q_2))$.

Now, let $M \in \mathrm{TS}(G;Q_2)$. Then, by the bijections in Subsection 2.3, there exists a unique PIM P_{ψ_i} of $k\overline{N}_G(Q_2)$, with $1 \leq i \leq p-1$, such that

$$\chi_{\widehat{M}} = \mathrm{Ind}_{\mathcal{F}\rtimes\langle h^{p+1}\rangle}^{G}\,\mathrm{Inf}_{(\mathcal{F}\rtimes\langle h^{p+1}\rangle)/\langle x\rangle}^{\mathcal{F}\rtimes\langle h^{p+1}\rangle}(\Phi_{\psi_i})$$

and by Proposition 4.3(d) the ordinary character of P_{ψ_i} of $k\overline{N}_G(Q_2)$ is given by

$$\Phi_{\psi_i} = \theta_i + \theta_p\,.$$

By Proposition 2.5, the induced module

$$\mathrm{Ind}^G_{\mathcal{F}\rtimes\langle h^{p+1}\rangle} \mathrm{Inf}^{\mathcal{F}\rtimes\langle h^{p+1}\rangle}_{(\mathcal{F}\rtimes\langle h^{p+1}\rangle)/\langle x\rangle}(P_{\psi_i})$$

is indecomposable. Hence, the Green correspondent of $\mathrm{Inf}^{N_G(Q_2)}_{\overline{N}_G(Q_2)}(P_{\psi_i})$ is

$$f(\mathrm{Inf}^{N_G(Q_2)}_{\overline{N}_G(Q_2)}(P_{\psi_i})) = \mathrm{Ind}^G_{\mathcal{F}\rtimes\langle h^{p+1}\rangle} \mathrm{Inf}^{\mathcal{F}\rtimes\langle h^{p+1}\rangle}_{(\mathcal{F}\rtimes\langle h^{p+1}\rangle)/\langle x\rangle}(P_{\psi_i}).$$

Next, we compute the ordinary characters of these modules. First, note that inflation does not change the degree of a character. Then, by Frobenius reciprocity, we have

$$\langle \mathrm{Ind}^G_{\mathcal{F}\rtimes\langle h^{p+1}\rangle} \mathrm{Inf}^{\mathcal{F}\rtimes\langle h^{p+1}\rangle}_{(\mathcal{F}\rtimes\langle h^{p+1}\rangle)/\langle x\rangle}(\theta_p), \chi_j\rangle = \langle \mathrm{Inf}^{\mathcal{F}\rtimes\langle h^{p+1}\rangle}_{(\mathcal{F}\rtimes\langle h^{p+1}\rangle)/\langle x\rangle}(\theta_p), \mathrm{Res}^G_{\mathcal{F}\rtimes\langle h^{p+1}\rangle}(\chi_j)\rangle = 0$$

for all $1 \leq j \leq p^2 - 1$ as $\{\chi_1, \ldots, \chi_{p^2-1}\} = \mathrm{Lin}(G)$. Hence, it follows that

$$\mathrm{Ind}^G_{\mathcal{F}\rtimes\langle h^{p+1}\rangle} \mathrm{Inf}^{\mathcal{F}\rtimes\langle h^{p+1}\rangle}_{(\mathcal{F}\rtimes\langle h^{p+1}\rangle)/\langle x\rangle}(\theta_p) = \chi_{p^2}\,.$$

Again, by Frobenius reciprocity, we obtain for each integer $0 \leq a \leq p$ that

$$\langle \mathrm{Ind}^G_{\mathcal{F}\rtimes\langle h^{p+1}\rangle} \mathrm{Inf}^{\mathcal{F}\rtimes\langle h^{p+1}\rangle}_{(\mathcal{F}\rtimes\langle h^{p+1}\rangle)/\langle x\rangle}(\theta_i), \chi_{a(p-1)+i}\rangle = \langle \mathrm{Inf}^{\mathcal{F}\rtimes\langle h^{p+1}\rangle}_{(\mathcal{F}\rtimes\langle h^{p+1}\rangle)/\langle x\rangle}(\theta_i), \mathrm{Res}^G_{\mathcal{F}\rtimes\langle h^{p+1}\rangle}(\chi_{a(p-1)+i})\rangle.$$

By inspecting Table 2 and Table 3, we deduce that the last expression is equal to 1 for each integer $0 \leq a \leq p$. Indeed, on the one hand, for each $1 \leq u \leq p-1$ and every $1 \leq v \leq p-2$ we have

$$\theta_u(h^{v(p+1)}) = \omega^{(u-1)v} = \zeta^{(u-1)(p+1)v}$$

and on the other hand, for each $1 \leq u \leq p-1$ and every $1 \leq v \leq p-2$ we have

$$\chi_{a(p-1)+u}(h^{v(p+1)}) = \zeta^{(a(p-1)+u-1)\cdot v(p+1)} = \zeta^{(p^2-1)av}\zeta^{(u-1)(p+1)v} = 1 \cdot \zeta^{(u-1)(p+1)v},$$

by definition of ζ and ω (see Proposition 4.3). Since the character degree of

$$\mathrm{Ind}^G_{\mathcal{F}\rtimes\langle h^{p+1}\rangle} \mathrm{Inf}^{\mathcal{F}\rtimes\langle h^{p+1}\rangle}_{(\mathcal{F}\rtimes\langle h^{p+1}\rangle)/\langle x\rangle}(\Phi_{\psi_i})$$

is equal to

$$[G : \mathcal{F} \rtimes \langle h^{p+1}\rangle] \cdot p = (p+1) \cdot p = p^2 + p = (p^2 - 1) + (p+1),$$

we have already found all irreducible constituents of the induced characters

$$\mathrm{Ind}^G_{\mathcal{F}\rtimes\langle h^{p+1}\rangle} \mathrm{Inf}^{\mathcal{F}\rtimes\langle h^{p+1}\rangle}_{(\mathcal{F}\rtimes\langle h^{p+1}\rangle)/\langle x\rangle}(\Phi_{\psi_i})$$

and it is obvious from the respective character tables that $\mathrm{Inf}^{\mathcal{F}\rtimes\langle h^{p+1}\rangle}_{(\mathcal{F}\rtimes\langle h^{p+1}\rangle)/\langle x\rangle}(\Phi_{\psi_i})$ coincides with $\chi_{p^2} + \sum_{a=0}^{p} \chi_{a(p-1)+i}$ at all other elements of $\mathcal{F} \rtimes \langle h^{p+1}\rangle$. In total, it follows that

$$\mathrm{Ind}^G_{F\rtimes\langle h^{p+1}\rangle} \mathrm{Inf}^{\mathcal{F}\rtimes\langle h^{p+1}\rangle}_{(\mathcal{F}\rtimes\langle h^{p+1}\rangle)/\langle x\rangle}(\Phi_{\psi_i}) = \chi_{p^2} + \sum_{a=0}^{p} \chi_{a(p-1)+i}\,,$$

as claimed. Finally, we obtain the entries of $T_{2,1}$ in Table 6 by evaluating the obtained characters at the p'-conjugacy classes of G using Proposition 4.3(b).

The matrix $T_{3,1}$. By Lemma 2.4(c) we have

$$T_{3,1} = \left(\chi_{\widehat{M}}(s)\right)_{M\in\mathrm{TS}(G;Q_3), s\in[G]_{p'}}.$$

Now, Lemma 4.2 yields $N_G(Q_3) = G$ and $\overline{N}_G(Q_3) = G/Q_3 \cong \mathcal{H}$. Hence, by the bijections in Subsection 2.3, we have

$$\mathrm{TS}(G; Q_3) = \{\mathrm{Inf}_{\mathcal{H}}^G(P) \mid P \text{ PIM of } k\mathcal{H}\}.$$

Because $\mathcal{H}$ is a p'-group, the ordinary characters of the PIMs of $k\mathcal{H}$ are given by $\mathrm{Irr}(\mathcal{H})$. It follows that the ordinary characters $\chi_{\widehat{M}}$ of the trivial source modules $M \in \mathrm{TS}(G; Q_3)$ are precisely given by $\{\mathrm{Inf}_{G/Q_3}^G(\xi) \mid \xi \in \mathrm{Irr}(\mathcal{H})\}$. A set of representatives of the p'-conjugacy classes of $N_G(Q_3) = G$ is given by the elements of $\mathcal{H}$. Therefore, $T_{3,1} = X(\mathcal{H})$ and the values in $T_{3,1}$ in Table 6 are as claimed by Notation 2.1.

(c) The matrix $T_{2,2}$. As in part (b), we follow the order of the previously computed decomposition matrix such that the order of the rows of $T_{2,2}$ and the order of the rows of $T_{2,1}$ coincide. By Lemma 2.4(b), we have

$$T_{2,2} = \Phi_p(\overline{N}_G(Q_2)) = \mathrm{Dec}_p(\overline{N}_G(Q_2))^t \cdot X(\overline{N}_G(Q_2), p').$$

Hence, the values of $T_{2,2}$ in Table 7 are obtained from Table 3 and Table 5. Moreover, recall that in $T_{2,1}$ above, we sorted the modules in $\mathrm{TS}(G; Q_2)$ according to the labelling of the rows of $\Phi_p(\overline{N}_G(Q_2))$. It follows that the labellings of the rows of $T_{2,2}$ and $T_{2,1}$ coincide.

The matrix $T_{3,2}$. By definition, we have $T_{3,2} = \left[\tau_{Q_2,s}^G([M])\right]_{M \in \mathrm{TS}(G;Q_3), s \in [\overline{N}_G(Q_2)]_{p'}}$. So, let $M \in \mathrm{TS}(G; Q_3)$ and let $t \in [N_G(Q_2)]_{p'}$. By Lemma 2.3 we have

$$\tau_{Q_2,t}^G([M]) = \chi_{\widehat{M}}(g_p \cdot t)$$

for any $g_p \in Q_2 \setminus \{1\}$. Because $Q_2 \leq Q_3 = \mathcal{F} \trianglelefteq G$, the ordinary characters of the trivial source modules with vertex Q_3 have Q_2 in their kernels. Hence, we obtain

$$\chi_{\widehat{M}}(g_p \cdot t) = \chi_{\widehat{M}}(1_G \cdot t).$$

By Lemma 4.2, $[\overline{N}_G(Q_2)]_{p'} = \langle h^{p+1} \rangle$, therefore the values of $T_{3,2}$ in Table 7 are as claimed by Notation 2.1.

(d) The matrix $T_{3,3}$. Since $\overline{N}_G(Q_3) \cong \mathcal{H}$ which is a p'-group, by Lemma 2.4(b), we have

$$T_{3,3} = \Phi_p(\mathcal{H}) = \mathrm{Dec}_p(\mathcal{H})^t \cdot X(\mathcal{H}, p') = X(\mathcal{H}),$$

as claimed. Moreover, the labelling of the rows of $T_{3,3}$ coincides with the labelling of the rows of $T_{3,1}$ by our fixed labelling of $\mathrm{Irr}(\mathcal{H})$ given in Notation 2.1 and the bijections in Subsection 2.3.

□

Example 4.5. Let G be the Frobenius group $(C_3 \times C_3) \rtimes C_8$ of order 72 with maximal fusion pattern, i.e. G has precisely one conjugacy class of subgroups of order 3. It follows that we have 3 conjugacy classes of 3-subgroups of G, namely:

$$Q_1 = \{1\}, \quad Q_2 \cong C_3, \quad Q_3 \cong C_3 \times C_3.$$

Notice that G is isomorphic to the group labelled by [72, 39] in GAP's SmallGroups library, see [GAP]. The ordinary character table of G is as given in Table 9, where ζ_8 denotes a primitive 8-th root of unity.

	$1a$	$8a$	$4a$	$8b$	$2a$	$8c$	$4b$	$8d$	$3a$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	ζ_8	ζ_8^2	ζ_8^3	-1	$-\zeta_8$	$-\zeta_8^2$	$-\zeta_8^3$	1
χ_3	1	ζ_8^2	-1	$-\zeta_8^2$	1	ζ_8^2	-1	$-\zeta_8^2$	1
χ_4	1	ζ_8^3	$-\zeta_8^2$	ζ_8	-1	$-\zeta_8^3$	ζ_8^2	$-\zeta_8$	1
χ_5	1	-1	1	-1	1	-1	1	-1	1
χ_6	1	$-\zeta_8$	ζ_8^2	$-\zeta_8^3$	-1	ζ_8	$-\zeta_8^2$	ζ_8^3	1
χ_7	1	$-\zeta_8^2$	-1	ζ_8^2	1	$-\zeta_8^2$	-1	ζ_8^2	1
χ_8	1	$-\zeta_8^3$	$-\zeta_8^2$	$-\zeta_8$	-1	ζ_8^3	ζ_8^2	ζ_8	1
χ_9	8	0	0	0	0	0	0	0	-1

TABLE 9. Ordinary character table of $(C_3 \times C_3) \rtimes C_8$

The trivial source character table $\mathrm{Triv}_3(G)$ is as given in Table 10. Note that we label the columns of $\mathrm{Triv}_3(G)$ with $3'$-elements in N_i instead of $\overline{N}_i$ $(1 \leq i \leq 3)$.

Normalisers N_i	$N_1 \cong (C_3 \times C_3) \rtimes C_8$								$N_2 \cong (C_3 \times C_3) \rtimes C_2$		$N_3 \cong (C_3 \times C_3) \rtimes C_8$							
Representatives $n_j \in N_i$	$1a$	$8a$	$4a$	$8b$	$2a$	$8c$	$4b$	$8d$	$1a$	$2a$	$1a$	$8a$	$4a$	$8b$	$2a$	$8c$	$4b$	$8d$
$\chi_1 + \chi_9$	9	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
$\chi_2 + \chi_9$	9	ζ_8	ζ_8^2	ζ_8^3	-1	$-\zeta_8$	$-\zeta_8^2$	$-\zeta_8^3$	0	0	0	0	0	0	0	0	0	0
$\chi_3 + \chi_9$	9	ζ_8^2	-1	$-\zeta_8^2$	1	ζ_8^2	-1	$-\zeta_8^2$	0	0	0	0	0	0	0	0	0	0
$\chi_4 + \chi_9$	9	ζ_8^3	$-\zeta_8^2$	ζ_8	-1	$-\zeta_8^3$	ζ_8^2	$-\zeta_8$	0	0	0	0	0	0	0	0	0	0
$\chi_5 + \chi_9$	9	-1	1	-1	1	-1	1	-1	0	0	0	0	0	0	0	0	0	0
$\chi_6 + \chi_9$	9	$-\zeta_8$	ζ_8^2	$-\zeta_8^3$	-1	ζ_8	$-\zeta_8^2$	ζ_8^3	0	0	0	0	0	0	0	0	0	0
$\chi_7 + \chi_9$	9	$-\zeta_8^2$	-1	ζ_8^2	1	$-\zeta_8^2$	-1	ζ_8^2	0	0	0	0	0	0	0	0	0	0
$\chi_8 + \chi_9$	9	$-\zeta_8^3$	$-\zeta_8^2$	$-\zeta_8$	-1	ζ_8^3	ζ_8^2	ζ_8	0	0	0	0	0	0	0	0	0	0
$\chi_1 + \chi_3 + \chi_5 + \chi_7 + \chi_9$	12	0	0	0	4	0	0	0	3	1	0	0	0	0	0	0	0	0
$\chi_2 + \chi_4 + \chi_6 + \chi_8 + \chi_9$	12	0	0	0	-4	0	0	0	3	-1	0	0	0	0	0	0	0	0
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	ζ_8	ζ_8^2	ζ_8^3	-1	$-\zeta_8$	$-\zeta_8^2$	$-\zeta_8^3$	1	-1	1	ζ_8	ζ_8^2	ζ_8^3	-1	$-\zeta_8$	$-\zeta_8^2$	$-\zeta_8^3$
χ_3	1	ζ_8^2	-1	$-\zeta_8^2$	1	ζ_8^2	-1	$-\zeta_8^2$	1	1	1	ζ_8^2	-1	$-\zeta_8^2$	1	ζ_8^2	-1	$-\zeta_8^2$
χ_4	1	ζ_8^3	$-\zeta_8^2$	ζ_8	-1	$-\zeta_8^3$	ζ_8^2	$-\zeta_8$	1	-1	1	ζ_8^3	$-\zeta_8^2$	ζ_8	-1	$-\zeta_8^3$	ζ_8^2	$-\zeta_8$
χ_5	1	-1	1	-1	1	-1	1	-1	1	1	1	-1	1	-1	1	-1	1	-1
χ_6	1	$-\zeta_8$	ζ_8^2	$-\zeta_8^3$	-1	ζ_8	$-\zeta_8^2$	ζ_8^3	1	-1	1	$-\zeta_8$	ζ_8^2	$-\zeta_8^3$	-1	ζ_8	$-\zeta_8^2$	ζ_8^3
χ_7	1	$-\zeta_8^2$	-1	ζ_8^2	1	$-\zeta_8^2$	-1	ζ_8^2	1	1	1	$-\zeta_8^2$	-1	ζ_8^2	1	$-\zeta_8^2$	-1	ζ_8^2
χ_8	1	$-\zeta_8^3$	$-\zeta_8^2$	$-\zeta_8$	-1	ζ_8^3	ζ_8^2	ζ_8	1	-1	1	$-\zeta_8^3$	$-\zeta_8^2$	$-\zeta_8$	-1	ζ_8^3	ζ_8^2	ζ_8

TABLE 10. Trivial source character table of $(C_3 \times C_3) \rtimes C_8$ at $p = 3$

5. The minimal fusion case

We now turn to the computation of the trivial source character tables of metabelian Frobenius groups with Frobenius kernel $F \cong C_p \times C_p$ and cyclic Frobenius complement $H \cong C_m$ such that no two conjugacy classes of p-subgroups of G of order p are G-conjugate. Recall from Proposition 3.7(b) that $|H|$ divides $p-1$ and that H acts on F by raising each element to a power of itself in this case.

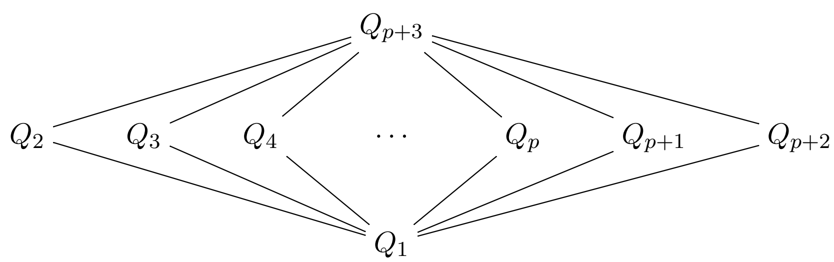
Notation 5.1. Throughout this section, we adopt the following notation. We choose a generator h of C_{p-1} and let $H := \langle h^{a(m)} \rangle$, where $a(m) := \frac{p-1}{m}$. We let $\{x, y\}$ be a set of generators for F. We choose the following set of representatives of the conjugacy classes of G:

$$\{x^i y^j \mid 1 \leq i, j \leq p\} \cup \{h^{a(m)}, h^{2a(m)}, \dots, h^{(m-1)a(m)}\}.$$

Moreover, by Proposition 3.7, we can choose the following set of representatives for the G-conjugacy classes of p-subgroups of G:

$$\begin{aligned} Q_1 &:= \{1\}, \\ Q_i &:= \langle x \cdot y^{i-2} \rangle \quad (2 \leq i \leq p+1), \\ Q_{p+2} &:= \langle y \rangle, \\ Q_{p+3} &:= F. \end{aligned}$$

Then, up to G-conjugation, the lattice of subgroups of G of order p is as given below.



As in Proposition 3.4, we let $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_{m+(p+1)\cdot a(m)}\}$ where for each $1 \leq i \leq m$ we set $\chi_i := \mathrm{Inf}_{G/F}^{G}(\xi_i)$ with $\xi_i \in \mathrm{Irr}(H)$ as defined in Notation 2.1, and

$$\{\chi_{m+1}, \dots, \chi_{m+(p+1)\cdot a(m)}\} = \{\mathrm{Ind}_F^G(\nu) \mid \nu \in T\}$$

where T is a set of representatives for the conjugation action of G on $\mathrm{Irr}(F) \setminus \{1_F\}$.

Lemma 5.2. *With the notation of Notation 5.1, for each $1 \leq j \leq p+3$ the following assertions hold:*

(a) $N_G(Q_j) = G$ *and* $\overline{N}_G(Q_j) = G/Q_j$*;*

(b) $\overline{N}_G(Q_j)$ *is a Frobenius group with Frobenius complement* $H \cong HQ_j/Q_j$ *and Frobenius kernel* F/Q_j*.*

Proof. (a) As no two distinct p-subgroups of G of order p are G-conjugate, it follows that $N_G(Q_j) = G$ for each $1 \leq j \leq p+3$. The second claim is then immediate.

(b) As G is a Frobenius group with respect to H, $Q_j \lneqq F$ and $Q_j \lneqq G$, the assertion follows from the definition. $\square$

Notation 5.3. For an arbitrary but fixed $a \in \{2, \dots, p+2\}$, we introduce the following notation. As in Proposition 3.4, we let $\mathrm{Irr}(\overline{N}_G(Q_a)) = \{\theta_1, \dots, \theta_{m+\frac{p-1}{m}}\}$ where for each $1 \leq i \leq m$ we set $\theta_i := \mathrm{Inf}^{G/Q_a}_{(G/Q_a)/(F/Q_a)}(\xi_i)$ with $\xi_i \in \mathrm{Irr}(H)$ as defined in Notation 2.1, and

$$\{\theta_{m+1}, \dots, \theta_{m+\frac{p-1}{m}}\} = \{\mathrm{Ind}^{G/Q_a}_{F/Q_a}(\nu) \mid \nu \in V\}$$

where V is a set of representatives for the conjugation action of G/Q_a on $\mathrm{Irr}(F/Q_a) \setminus \{1_{F/Q_a}\}$.

Proposition 5.4. (a) *Setting $\varphi_i := \chi_i^\circ$ for each $1 \leq i \leq m$, then $\mathrm{IBr}_p(G) = \{\varphi_1, \dots, \varphi_m\}$ and $\mathrm{Dec}_p(G)$ is as given in Table 11.*

	φ_1	φ_2	$\cdots$	φ_{m-1}	φ_m
χ_1	1	0	$\cdots$	0	0
χ_2	0	1	$\ddots$	$\vdots$	$\vdots$
χ_3	0	0	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	1	0
χ_m	0	0	$\cdots$	0	1
χ_{m+1}	1	1	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\chi_{m+(p+1)\cdot a(m)}$	1	1	$\cdots$	1	1

TABLE 11. p-decomposition matrix of G

(b) *Let $a \in \{2, \dots, p+2\}$. Setting $\psi_i := \theta_i^\circ$ for each $1 \leq i \leq m$, then $\mathrm{IBr}_p(\overline{N}_G(Q_a)) = \{\psi_1, \dots, \psi_m\}$ and $\mathrm{Dec}_p(\overline{N}_G(Q_a))$ is as given in Table 12.*

	ψ_1	ψ_2	$\cdots$	ψ_{m-1}	ψ_m
θ_1	1	0	$\cdots$	0	0
θ_2	0	1	$\ddots$	$\vdots$	$\vdots$
θ_3	0	0	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	1	0
θ_m	0	0	$\cdots$	0	1
θ_{m+1}	1	1	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\theta_{m+\frac{p-1}{m}}$	1	1	$\cdots$	1	1

TABLE 12. p-decomposition matrix of $\overline{N}_G(Q_a)$

Proof. Both (a) and (b) are immediate from Proposition 3.4(c). □

Proposition 5.5. *Let G be a metableian Frobenius group with cyclic Frobenius complement H of order m and elementary abelian Frobenius kernel F of order p^2 such that the number of G-conjugacy classes of subgroups of G of order p equals $p+1$. Then, with the notation of Notation 5.1, the following assertions hold.*

(a) *Let M and N be trivial source kG-modules with non-trivial cyclic vertices. Then $M \cong N$ if and only if $\chi_{\widehat{M}} = \chi_{\widehat{N}}$.*

(b) *As sets* $\operatorname{Irr}(G) \setminus \operatorname{Lin}(G) = \coprod_{2 \le i \le p+2} \operatorname{Inf}_{G/Q_i}^{G} \big(\operatorname{Irr}(G/Q_i) \setminus \operatorname{Lin}(G/Q_i)\big)$.

(c) *Let $2 \le i \le p+2$. Let $M \in \operatorname{TS}(G; Q_i)$. Then*

$$\chi_{\widehat{M}} = \lambda + \sum_{\chi \in \operatorname{Irr}(G)\setminus\operatorname{Lin}(G), Q_i \le Ker(\chi)} \chi$$

for some $\lambda \in \operatorname{Lin}(G)$.

Proof. (a) It is clear that isomorphic modules afford the same ordinary characters. We prove the sufficient condition by contraposition. So assume that $M \ncong N$. First, we suppose that M and N have a common vertex Q_i for some $2 \le i \le p+2$.

We know from Lemma 5.2 that $\overline{N}_G(Q_i) = G/Q_i$ is a Frobenius group with Frobenius complement $H = Q_iH/Q_i$ and Frobenius kernel F/Q_i. It follows from Proposition 5.4(b) that for $1 \le u \le m$ the ordinary characters of the PIMs P_{ψ_u} of $k\overline{N}_G(Q_i)$ are given by $\Phi_{\psi_u} = \theta_u + \sum_{v=1}^{\frac{p-1}{m}} \theta_{m+v}$. Let $\operatorname{Inf}_{G/Q_i}^{G}(P_{\psi_{u_1}})$ and $\operatorname{Inf}_{G/Q_i}^{G}(P_{\psi_{u_2}})$ be the Green correspondents of M and N, respectively, where $1 \le u_1 \ne u_2 \le m$. Hence,

$$\chi_{\widehat{M}} = \operatorname{Inf}_{G/Q_i}^{G}(\Phi_{\psi_{u_1}}) \ne \operatorname{Inf}_{G/Q_i}^{G}(\Phi_{\psi_{u_2}}) = \chi_{\widehat{N}}.$$

We can now assume that M has vertex Q_i and N has vertex Q_j with $2 \le i \ne j \le p+2$. Hence, we can assume that

$$\chi_{\widehat{M}} = \operatorname{Inf}_{G/Q_i}^{G}(\Phi_{\psi_u}) = \operatorname{Inf}_{G/Q_i}^{G}(\theta_u + \sum_{v=1}^{\frac{p-1}{m}} \theta_{m+v})$$

for some $1 \le u \le m$. It follows from Proposition 2.5 that M and N correspond to simple kH-modules S and T, namely:

$$\begin{aligned} M &\cong \operatorname{Ind}_{N_G(Q_i)}^{G} \operatorname{Inf}_{\overline{N}_G(Q_i)}^{N_G(Q_i)} \operatorname{Ind}_{Q_iH/Q_i}^{\overline{N}_G(Q_i)}(S) \\ &= \operatorname{Ind}_{G}^{G} \operatorname{Inf}_{G/Q_i}^{G} \operatorname{Ind}_{Q_iH/Q_i}^{G/Q_i}(S) \\ &= \operatorname{Inf}_{G/Q_i}^{G} \operatorname{Ind}_{Q_iH/Q_i}^{G/Q_i}(S) = \operatorname{Ind}_{Q_iH}^{G} \operatorname{Inf}_{Q_iH/Q_i}^{Q_iH}(S) \end{aligned}$$

and analogously $N \cong \operatorname{Ind}_{Q_jH}^{G} \operatorname{Inf}_{Q_jH/Q_j}^{Q_jH}(T)$. We set $I := Q_i \rtimes H$, $J := Q_j \rtimes H$, $S_0 := \operatorname{Inf}_{Q_iH/Q_i}^{Q_iH}(S)$, and $T_0 := \operatorname{Inf}_{Q_jH/Q_j}^{Q_jH}(T)$. Hence, we obtain

$$\begin{aligned} \langle \chi_{\widehat{M}}, \chi_{\widehat{N}} \rangle &= \langle \operatorname{Ind}_{Q_i \rtimes H}^{G}(\chi_{\widehat{S_0}}), \operatorname{Ind}_{Q_j \rtimes H}^{G}(\chi_{\widehat{T_0}}) \rangle \\ &= \langle \chi_{\widehat{S_0}}, \operatorname{Res}_{Q_i \rtimes H}^{G}(\operatorname{Ind}_{Q_j \rtimes H}^{G}(\chi_{\widehat{T_0}})) \rangle \\ &= \langle \chi_{\widehat{S_0}}, \sum_{s \in I\setminus G/J} \operatorname{Ind}_{s^{-1}Js\cap I}^{I}(\operatorname{Res}_{s^{-1}Js\cap I}^{s^{-1}Js}({}^{s}\chi_{\widehat{T_0}})) \rangle, \end{aligned}$$

where the first equation holds by Proposition 2.5, the second equation holds by Frobenius reciprocity, and the last equation is true due to the Mackey formula. The set of double cosets $I \setminus G/J$ contains only one element because H acts on the abelian group F by raising each element to a power of itself. Therefore,

$$\begin{aligned}\langle \chi_{\widehat{M}}, \chi_{\widehat{N}} \rangle &= \langle \chi_{\widehat{S_0}}, \mathrm{Ind}^I_{J\cap I}(\mathrm{Res}^J_{J\cap I}(\chi_{\widehat{T_0}})) \rangle \\ &= \langle \chi_{\widehat{S_0}}, \mathrm{Ind}^I_H(\mathrm{Res}^J_H(\chi_{\widehat{T_0}})) \\ &= \langle \chi_{\widehat{S_0}}, \mathrm{Ind}^I_H(\chi_{\widehat{T}}) \rangle \leq 1,\end{aligned}$$

where the inequality holds due to the following argument. As H is a p'-group, the simple kH-module T is projective. Hence, also the kI-module $W := \mathrm{Ind}^I_H(T)$ is projective. Since the k-dimension of W equals $[I : H] \cdot 1 = \frac{|Q_i|\cdot|H|}{|H|} \cdot 1 = p$, the kI-module W is indecomposable, hence a PIM of kI. It follows from the decomposition matrix $\mathrm{Dec}_p(I)$ of I which we know from Proposition 3.4 that $\chi_{\widehat{W}}$ has only one linear character as a constituent. This is a contradiction, as the character $\chi_{\widehat{M}}$ has at least two irreducible constituents such that we would have to have $\langle \chi_{\widehat{M}}, \chi_{\widehat{N}} \rangle \geq 2$ if we really had $\chi_{\widehat{M}} = \chi_{\widehat{N}}$.

(b) Let $2 \leq i, j \leq p+2$ with $i \neq j$. We claim that the intersection

$$\{\mathrm{Inf}^G_{G/Q_i}(\mu) \mid \mu \in \mathrm{Irr}(G/Q_i) \setminus \mathrm{Lin}(G/Q_i)\} \cap \{\mathrm{Inf}^G_{G/Q_j}(\nu) \mid \nu \in \mathrm{Irr}(G/Q_j) \setminus \mathrm{Lin}(G/Q_j)\}$$

is empty. Assume the contrary and let γ be an element of this intersection. Note that γ is non-linear. We know from the proof of part (a) that

$$\{\chi_{\widehat{M}} \mid M \in \mathrm{TS}(G; Q_i)\} = \{\mathrm{Inf}^G_{G/Q_i}(\mathrm{Inf}^{G/Q_i}_H(\lambda) + \sigma_1) \mid \lambda \in \mathrm{Irr}(H)\},$$

where $\sigma_1 := \sum_{\alpha \in \mathrm{Irr}(G/Q_i)\setminus \mathrm{Lin}(G/Q_i)} \alpha$ and H is identified with $(G/Q_i)/(F/Q_i)$. Moreover, the set

$$\{\chi_{\widehat{N}} \mid N \in \mathrm{TS}(G; Q_j)\} = \{\mathrm{Inf}^G_{G/Q_j}(\mathrm{Inf}^{G/Q_j}_H(\widetilde{\lambda}) + \sigma_2) \mid \widetilde{\lambda} \in \mathrm{Irr}(H)\},$$

where $\sigma_2 := \sum_{\beta \in \mathrm{Irr}(G/Q_j)\setminus \mathrm{Lin}(G/Q_j)} \beta$. Specialising to $\lambda = \widetilde{\lambda} = 1_H$, we deduce that the scalar product

$$\langle \mathrm{Inf}^G_{G/Q_i}(1_{G/Q_i} + \sigma_1), \mathrm{Inf}^G_{G/Q_j}(1_{G/Q_i} + \sigma_2) \rangle \geq 2,$$

as γ is a non-linear constituent of both inflated characters. This is a contradiction to the proof of part (a).

Next, we see that Notation 5.3 yields

$$|\{\mathrm{Inf}^G_{G/Q_i}(\mu) \mid \mu \in \mathrm{Irr}(G/Q_i) \setminus \mathrm{Lin}(G/Q_i)\}| = \frac{p-1}{m}$$

for each $2 \leq i \leq p+2$. Hence, it follows that

$$\coprod_{2\leq i\leq p+2} \{\mathrm{Inf}^G_{G/Q_i}(\mu) \mid \mu \in \mathrm{Irr}(G/Q_i) \setminus \mathrm{Lin}(G/Q_i)\} = \mathrm{Irr}(G) \setminus \mathrm{Lin}(G),$$

as $|\mathrm{Irr}(G) \setminus \mathrm{Lin}(G)| = \frac{p^2-1}{m} = \frac{p-1}{m} \cdot (p+1)$.

(c) As in the proof of part (b), we know from part (a) that

$$\{\chi_{\widehat{M}} \mid M \in \mathrm{TS}(G; Q_i)\} = \{\mathrm{Inf}^G_{G/Q_i}(\mathrm{Inf}^{G/Q_i}_H(\lambda) + \sigma_1) \mid \lambda \in \mathrm{Irr}(H)\},$$

where $\sigma_1 := \sum_{\alpha \in \mathrm{Irr}(G/Q_i)\setminus \mathrm{Lin}(G/Q_i)} \alpha$. As G is a Frobenius group, we have

$$\mathrm{Lin}(G) = \{\mathrm{Inf}^G_{G/Q_i}(\mathrm{Inf}^{G/Q_i}_H(\xi)) \mid \xi \in \mathrm{Irr}(H)\}.$$

Moreover, we know from part (b) that

$$\sum_{\chi \in \mathrm{Irr}(G)\setminus \mathrm{Lin}(G), Q_i \leq Ker(\chi)} \chi = \sum_{\alpha \in \mathrm{Irr}(G/Q_i)\setminus \mathrm{Lin}(G/Q_i)} \mathrm{Inf}_{G/Q_i}^{G}(\alpha).$$

The assertion follows.

□

Theorem 5.6. *Let G be a metableian Frobenius group with cyclic complement H of order m dividing $p-1$ and Frobenius kernel $F \cong C_p \times C_p$ such that the number of G-conjugacy classes of subgroups of G of order p is precisely $p+1$. Then, the trivial source character table $\mathrm{Triv}_p(G)$ seen as a block matrix is as given in Table 13. For all $1 \leq i \leq p+3$, we choose the labelling of the columns of the i-th block column of $\mathrm{Triv}_p(G)$ precisely as the labelling of the columns of $X(H)$. For each $1 \leq i \leq p+3$, we choose the labelling of the rows of the i-th block row of $\mathrm{Triv}_p(G)$ as $\mathrm{Ind}_{Q_i \rtimes H}^{G}(\mathrm{Inf}_{H}^{Q_i \rtimes H}(\lambda_j))$, where $1 \leq j \leq m$ and λ_j denotes the character labelling the j-th row of $X(H)$. Then, the following assertions hold:*

(a) $T_{i,j} = \mathbf{0}$ *for every* $2 \leq j < i \leq p+2$ *and for every* $1 \leq i < j \leq p+3$*;*

(b) $T_{1,1} = X(H) + \begin{pmatrix} p^2-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p^2-1 & 0 & \cdots & 0 \end{pmatrix}$*;*

(c) $T_{2,1} = X(H) + \begin{pmatrix} p-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & \cdots & 0 \end{pmatrix}$*;*

(d) $T_{p+3,1} = X(H)$*.*

$T_{1,1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$T_{2,1}$	$T_{2,2} = T_{2,1}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$T_{3,1} = T_{2,1}$	$\mathbf{0}$	$T_{3,3} = T_{2,1}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$
$T_{p+2,1} = T_{2,1}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$T_{p+2,p+2} = T_{2,1}$	$\mathbf{0}$
$T_{p+3,1}$	$T_{p+3,2} = T_{p+3,1}$	$\cdots$	$\cdots$	$T_{p+3,p+2} = T_{p+3,1}$	$T_{p+3,p+3} = T_{p+3,1}$

TABLE 13. Trivial source character table $\mathrm{Triv}_p(G)$, seen as a block matrix

Proof. By Lemma 5.2, we have $N_G(Q_j) = G$ for each $1 \leq j \leq p+3$. As G is a semi-direct product of the normal p-subgroup F by the abelian p'-subgroup H, clearly H is a set of representatives of the p'-conjugacy classes of G. Hence, we can choose our labels of the columns of the block columns of $\mathrm{Triv}_p(G)$ as asserted.

(a) The assertion is immediate from Lemma 2.4(a).

(b) By Lemma 2.4(b), we have $T_{1,1} = \Phi_p(G)$. The labels of the rows of $T_{1,1}$ are the ordinary characters of the PIMs of kG and can be read off from the decomposition matrix in

Table 11, that is, for each $1 \leq i \leq m$ we have

$$\Phi_{\varphi_i} = \chi_i + \sum_{j=m+1}^{m+(p+1)\cdot a(m)} \chi_j.$$

In order to prove that

$$T_{1,1} = X(H) + \begin{pmatrix} p^2-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p^2-1 & 0 & \cdots & 0 \end{pmatrix},$$

it is only left to prove that $\chi_j(u) = 0$ for all $m+1 \leq j \leq m+(p+1)\cdot a(m)$ and for all $u \in G_{p'} \setminus \{1\}$. But this follows easily from the formula for induced characters and from the fact that conjugation does not change the order of a group element. Indeed, for any $y \in H \setminus \{1\}$ and for any $\nu \in \mathrm{Irr}(F)$, we have

$$(\mathrm{Ind}_F^G(\nu))(y) = \frac{1}{|F|} \sum_{x \in G} \dot{\nu}(xyx^{-1}) = 0.$$

(c) The matrices $T_{a,a}$ $(2 \leq a \leq p+2)$. By Lemma 5.2 the group $\overline{N}_G(Q_a) = G/Q_a$ is a Frobenius group with Frobenius complement H and Frobenius kernel F/Q_a. It follows from Proposition 5.4(b) that the ordinary characters of the PIMs P_{ψ_i} of $k\overline{N}_G(Q_a)$ are given by

$$\Phi_{\psi_i} = \theta_i + \sum_{j=1}^{\frac{p-1}{m}} \theta_{m+j}$$

for all $1 \leq i \leq m$. As every non-linear ordinary irreducible character of G/Q_a is induced from a linear character of F/Q_a, their degrees are $[G/Q_a : F/Q_a]\cdot 1 = [G:F] = |H| = m$. Therefore, $\deg(\Phi_{\psi_i}) = 1 + \frac{p-1}{m} \cdot m = 1 + p - 1 = p$ for all $1 \leq i \leq m$. Using the formula for induced characters, we see that all non-linear constituents of Φ_{ψ_i} evaluate to 0 at g if $g \in H \setminus \{1\}$. Moreover, all linear characters of $\overline{N}_G(Q_a) = G/Q_a$ are given by inflations of linear characters of H. As $[G/Q_a]_{p'} = H$, it follows that

$$T_{a,a} = X(H) + \begin{pmatrix} p-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & \cdots & 0 \end{pmatrix},$$

as claimed.

The matrices $T_{a,1}$ $(2 \leq a \leq p+2)$. By Lemma 2.4(c), we have

$$T_{a,1} = \big(\chi_{\widehat{M}}(s)\big)_{M \in \mathrm{TS}(G;Q_a), s \in [G]_{p'}}.$$

Let $M \in TS(G; Q_a)$. By the bijections in Subsection 2.3, we have

$$\chi_{\widehat{M}} = \mathrm{Ind}_{N_G(Q_a)}^G \mathrm{Inf}_{N_G(Q_a)/Q_a}^{N_G(Q_a)}(\Phi_{\psi_i}) = \mathrm{Ind}_G^G \mathrm{Inf}_{G/Q_a}^G(\Phi_{\psi_i}) = \mathrm{Inf}_{G/Q_a}^G(\Phi_{\psi_i})$$

for a unique $i \in \{1, \ldots, m\}$. As $[G]_{p'} = H$, we have $T_{a,1} = T_{a,a} = T_{2,1}$ for all $2 \leq a \leq p+2$, as asserted. The ordinary characters $\mathrm{Inf}_{G/Q_a}^G(\Phi_{\psi_i})$ are described in details in Proposition 5.5 and they coincide with the given labelling of the rows of $T_{a,1}$.

(d) The matrices $T_{p+3,b}$ $(1 \leq b \leq p+3)$. By Lemma 2.4(b), we have

$$T_{p+3,p+3} = \Phi_p(\overline{N}_G(Q_{p+3})) = \mathrm{Dec}_p(\overline{N}_G(Q_{p+3}))^t \cdot X(\overline{N}_G(Q_{p+3}), p').$$

As $\overline{N}_G(Q_{p+3}) = G/F \cong H$ is a cyclic p'-group, the PIMs P_i $(1 \leq i \leq m)$ of $k[G/F] \cong kH$ are, up to isomorphism, precisely the simple kH-modules. Hence, $\mathrm{Dec}_p(\overline{N}_G(Q_{p+3}))$ is equal to the identity matrix of size $m \times m$ and $X(\overline{N}_G(Q_{p+3}), p') = X(H)$. As H is a cyclic p'-group, its ordinary characters follow from Notation 2.1 and $\chi_{\widehat{P_i}}$ corresponds to the i-th row of $X(H)$ for all $1 \leq i \leq m$. This determines $T_{p+3,p+3}$.

Now we compute $T_{p+3,1}$. By the bijections in Subsection 2.3 and by Proposition 2.5, the ordinary character of the Green correspondent of the $k[\overline{N}_G(Q_{p+3})]$-module P_i is given by

$$\mathrm{Ind}_{N_G(Q_{p+3})}^{G} \mathrm{Inf}_{N_G(Q_{p+3})/Q_{p+3}}^{N_G(Q_{p+3})}(\chi_{\widehat{P_i}}) = \mathrm{Ind}_G^G \mathrm{Inf}_{G/Q_{p+3}}^{G}(\chi_{\widehat{P_i}}) = \mathrm{Inf}_{G/F}^{G}(\chi_{\widehat{P_i}}) = \chi_i.$$

As $[G]_{p'} = H$, the matrix $T_{p+3,1}$ is as asserted.

Finally, for $2 \leq b \leq p+2$, the entries of the matrix $T_{p+3,b}$ follow from Lemma 2.3. Let $Q_b = \langle t_b \rangle$ and let $s \in [G]_{p'}$. Then, for any $M \in \mathrm{TS}(G;F)$ we have

$$\tau_{\langle t_b \rangle}^{G}(s) = \chi_{\widehat{M}}(t_b \cdot s) = \chi_{\widehat{M}}(s),$$

since $F \trianglelefteq G$ and $F \leq \mathrm{Ker}(\lambda)$ for all $\lambda \in \mathrm{Lin}(G)$.

□

Example 5.7. Let G be the Frobenius group $(C_5 \times C_5) \rtimes C_4$ of order 100 with minimal fusion pattern, i.e. G has precisely 6 distinct conjugacy classes of subgroups of order 5. It follows that we have 8 conjugacy classes of 5-subgroups of G, namely:

$$Q_1 = \{1\}, Q_2 \cong Q_3 \cong Q_4 \cong Q_5 \cong Q_6 \cong Q_7 \cong C_5, Q_8 \cong C_5 \times C_5.$$

Notice that G is isomorphic to the group labelled by [100, 11] in GAP's SmallGroups library, see [GAP]. The ordinary character table of G is as given in Table 14, where ζ_4 denotes a primitive 4-th root of unity.

	$1a$	$4a$	$2a$	$4b$	$5a$	$5b$	$5c$	$5d$	$5e$	$5f$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	ζ_4	-1	$-\zeta_4$	1	1	1	1	1	1
χ_3	1	-1	1	-1	1	1	1	1	1	1
χ_4	1	$-\zeta_4$	-1	ζ_4	1	1	1	1	1	1
χ_5	4	0	0	0	4	-1	-1	-1	-1	-1
χ_6	4	0	0	0	-1	4	-1	-1	-1	-1
χ_7	4	0	0	0	-1	-1	4	-1	-1	-1
χ_8	4	0	0	0	-1	-1	-1	4	-1	-1
χ_9	4	0	0	0	-1	-1	-1	-1	4	-1
χ_{10}	4	0	0	0	-1	-1	-1	-1	-1	4

TABLE 14. Ordinary character table of $(C_5 \times C_5) \rtimes C_4$

The trivial source character table $\mathrm{Triv}_5(G)$ is as given in Table 15. Note that we label the columns of $\mathrm{Triv}_5(G)$ with $5'$-elements in N_i instead of $\overline{N}_i$ $(1 \leq i \leq 8)$.

Normalisers N_i	$N_1 \cong (C_5 \times C_5) \rtimes C_4$				$N_2 \cong (C_5 \times C_5) \rtimes C_4$				$N_3 \cong (C_5 \times C_5) \rtimes C_4$				$N_4 \cong (C_5 \times C_5) \rtimes C_4$				$N_5 \cong (C_5 \times C_5) \rtimes C_4$				$N_6 \cong (C_5 \times C_5) \rtimes C_4$				$N_7 \cong (C_5 \times C_5) \rtimes C_4$				$N_8 \cong (C_5 \times C_5) \rtimes C_4$			
Representatives $n_j \in N_i$	$1a$	$4a$	$2a$	$4b$	$1a$	$4a$	$2a$	$4b$	$1a$	$4a$	$2a$	$4b$	$1a$	$4a$	$2a$	$4b$	$1a$	$4a$	$2a$	$4b$	$1a$	$4a$	$2a$	$4b$	$1a$	$4a$	$2a$	$4b$	$1a$	$4a$	$2a$	$4b$
$\chi_1+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}$	25	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_2+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}$	25	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_3+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}$	25	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_4+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}$	25	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_1+\chi_5$	5	1	1	1	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_2+\chi_5$	5	ζ_4	-1	$-\zeta_4$	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_3+\chi_5$	5	-1	1	-1	5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_4+\chi_5$	5	$-\zeta_4$	-1	ζ_4	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_1+\chi_6$	5	1	1	1	0	0	0	0	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_2+\chi_6$	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_3+\chi_6$	5	-1	1	-1	0	0	0	0	5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_4+\chi_6$	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_1+\chi_7$	5	1	1	1	0	0	0	0	0	0	0	0	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_2+\chi_7$	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_3+\chi_7$	5	-1	1	-1	0	0	0	0	0	0	0	0	5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_4+\chi_7$	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_1+\chi_8$	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_2+\chi_8$	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_3+\chi_8$	5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_4+\chi_8$	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_1+\chi_9$	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	1	1	1	0	0	0	0	0	0	0	0
$\chi_2+\chi_9$	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0
$\chi_3+\chi_9$	5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	-1	1	-1	0	0	0	0	0	0	0	0
$\chi_4+\chi_9$	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0
$\chi_1+\chi_{10}$	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	1	1	1	0	0	0	0
$\chi_2+\chi_{10}$	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	ζ_4	-1	$-\zeta_4$	0	0	0	0
$\chi_3+\chi_{10}$	5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	-1	1	-1	0	0	0	0
$\chi_4+\chi_{10}$	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	$-\zeta_4$	-1	ζ_4	0	0	0	0
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$
χ_3	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4

TABLE 15. Trivial source character table of $(C_5 \times C_5) \rtimes C_4$ at $p = 5$

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