Trivial source character tables of small finite groups

Contact Information:
Bernhard Boehmler, M.Sc.
AGAG, Faculty of Mathematics
TU Kaiserslautern
Email: boehmler@mathematik.uni-kl.de

Abstract

Trivial source modules, also called p-permutation modules, arise naturally in the representation theory of finite groups. They are, by definition, the indecomposable direct summands of the permutation modules. Trivial source modules are building pieces for Puig equivalences, endo-permutation source equivalences, p-permutation equivalences, and splendid Rickard equivalences of blocks. They are also relevant to Alperin's weight conjecture. For example, a simple module is a weight module if and only if it is a trivial source module.

In order to do calculations with trivial source modules the ordinary characters of their lifts from positive characteristic p to characteristic zero are of particular interest. The "trivial source character tables" or "species tables" collect information about the character values of trivial source kG-modules with all possible vertices and those of their Brauer constructions. They also implicitly contain information about decomposition matrices.

General assumptions

- G is a finite group and p is a prime dividing $|G|$
- $(k, \mathcal{O}, K)$ is a p-modular system which is large enough for all quotients of all subgroups of G
- all modules considered are finite-dimensional
- trivial source modules are assumed to be indecomposable

Trivial source modules

Every **trivial source module** occurs as a direct summand of $k\uparrow_P^G$ for some p-subgroup P of G.

All these modules have trivial sources, hence their name.

For an arbitrary (but fixed) p there are only finitely many trivial source kG-modules up to isomorphism, whereas there are in general infinitely many indecomposable kG-modules, up to isomorphism.

Trivial source modules are also called **p-permutation modules** since
a non-zero kG-module V is a trivial source module if and only if V has a k-basis on which a Sylow p-subgroup of G acts as a permutation group.

Essential property:
Trivial source kG-modules lift in a unique way to trivial source $\mathcal{O}G$-lattices wherefore we can consider the K-characters of these lifts.

Trivial source character tables

In ordinary representation theory the character table collects the values of characters of group representations at conjugacy classes. Analogously, the **trivial source character table** collects the character values of the Brauer constructions at p'-elements of the normalisers of the p-subgroups of G up to conjugacy in G. In particular:

- ◇ each line corresponds to a trivial source kG-module
- ◇ the first column lists the dimensions of the kG-modules
- ◇ the first block column lists the values of the K-characters (of the lifts from characteristic p to characteristic 0 of the trivial source kG-modules) at the p-regular classes of G
- ◇ the first block line corresponds to the projective indecomposable kG-modules
- ◇ the entries of the diagonal blocks correspond to the Brauer character values of the PIMs of the factor groups $N_G(P_i)/P_i$, where the P_i run over the p-subgroups of G up to conjugacy in G
- ◇ entries in off-diagonal blocks are obtained by computing character values of the Brauer constructions at p'-elements of the respective normalisers wherefore all entries are zero above the main block diagonal, when ordering the vertices according to their orders

Example: $G = S_5,\ p = 2$

In the following trivial source character table we label the K-characters as in the GAP character table library *CTblLib*.

$G = S_5$		Q	$\langle 1 \rangle$			C_2		C_2	V_4		V_4	C_4	D_8
$p = 2$		N	S_5			D_{12}		D_8	S_4		D_8	D_8	D_8
		$\overline{N}$	S_5			S_3		V_4	S_3		C_2	C_2	$\langle 1 \rangle$
	P	$\overline{n} \in \overline{N}$	$\overline{1}$	$\overline{(123)}$	$\overline{(12345)}$	$\overline{1}$	$\overline{(345)}$	$\overline{1}$	$\overline{1}$	$\overline{(234)}$	$\overline{1}$	$\overline{1}$	$\overline{1}$
$\chi_1 + \chi_2 + 2\chi_3 + \chi_6 + \chi_7$	1	$\langle 1 \rangle$	24	0	4	0	0	0	0	0	0	0	0
$\chi_4 + \chi_5$	2		8	2	-2	0	0	0	0	0	0	0	0
$\chi_3 + \chi_6 + \chi_7$	3		16	-2	1	0	0	0	0	0	0	0	0
$\chi_1 + \chi_3 + \chi_6$	4	C_2	12	0	2	2	2	0	0	0	0	0	0
χ_4	5		4	1	-1	2	-1	0	0	0	0	0	0
$\chi_1 + \chi_2 + \chi_6 + \chi_7$	6	C_2	12	0	2	0	0	4	0	0	0	0	0
$\chi_1 + \chi_2$	7	V_4	2	2	2	0	0	2	2	2	0	0	0
$\chi_6 + \chi_7$	8		10	-2	0	0	0	2	2	-1	0	0	0
$\chi_1 + \chi_6$	9	V_4	6	0	1	2	2	2	0	0	2	0	0
$\chi_1 + \chi_7$	10	C_4	6	0	1	0	0	2	0	0	0	2	0
χ_1	11	D_8	1	1	1	1	1	1	1	1	1	1	1

Every group listed in column P is one representative of the p-subgroups of G up to conjugacy in G. They are the vertices of the trivial source kG-modules.

The row labelled by Q has the same groups as entries.

Every element in the orange row is a representative of the p'-classes of $\overline{N}$.

The groups next to N are the normalisers in G of the groups in the row Q.

Main objectives of the project

- Investigate and determine trivial source character tables of finite groups.
- Contribute to the development of a database of trivial source character tables.
- Publish this database in electronic form.

Methods applied

With the aid of the computer algebra systems GAP, MAGMA and SAGE we wrote an algorithm that calculates the trivial source character table for a given finite group G and a prime p dividing $|G|$ (provided that the ordinary character table and the decomposition matrix are known).

In order to verify the results, we use the following theory:

- the Green correspondence
- the Brauer construction
- the theory of blocks of finite groups
- p-Kostka numbers and the theory of Specht modules (for $G = S_n$)

Results

For example, we have obtained the trivial source character tables of the following groups:

- $G = A_n$ for $n \in \{3, \ldots, 7\}$ for arbitrary p dividing $|G|$ (computationally)
- $G = S_n$ for $n \in \{3, \ldots, 7\}$ for arbitrary p dividing $|G|$ (computationally)
- the infinite family of dihedral groups $G = \mathrm{D}_{2n}$, where $2n = 4v$, v odd, for $p = 2$ (both computationally and theoretically)
- the unitary group $U_3(3)$ for $p = 3$ (computationally)
- the groups $PSL_2(q)$ for $p = 3$ and $q \in \{5, 11, 13, 19\}$ (computationally)
- the case $G = M_{11}, p = 2$ (computationally)
- the case $G = M_{11}, p = 3$ (computationally, we recover [3, §4.10])

References

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